Kinetic Brownian motion on the diffeomorphism group of a closed Riemannian manifold

J. Angst, I. Bailleul and P. Perruchaud

Abstract. We define and study a new model for random perturbations of fluid dynamics, namely kinetic Brownian motion on the diffeomorphism group of a general closed Riemannian manifold. Using rough paths techniques and extending the notion of stochastic development to the infinite-dimensional setting of Hilbertian manifolds, we rigorously establish that this model provides an interpolation between the hydrodynamic flow of a fluid and a Brownian-like flow.

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Keywords. Diffeomorphism group; EPDiff; Stochastic Euler equation; Cartan development; Brownian flow.

Contents

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1. Introduction

We introduce a new model for random perturbations of fluid dynamics. In a nutshell, it is based on two main ingredients: on the one hand Arnold's approach to fluid mechanics, which rewrites the motion of an incompressible inviscid fluid as a geodesic in some infinite-dimensional configuration space; and on the other hand kinetic Brownian motion, a stochastic process interpolating between the geodesic flow and Brownian motion. An important aspect of our approach is the use of rough path theory, which makes our constructions easy to adapt to similar problems, and provides a sturdy framework to study the noise dependency of the model.

1.1. Motivations

The fundamental idea behind Arnold's approach to fluid mechanics is to see the physics of a fluid as an infinite-dimensional version of a n-body problem, as it is usually described in classical mechanics. In this simple situation, the configuration of n bodies living in some space M can be described as a function from $\{1,\dots,n\}$ to M. In general, the dynamics evolves according to the Hamilton equations associated to some Hamiltonian over the cotangent of this configuration space M^n. For the case of non-interacting particles, this Hamiltonian is the kinetic energy of the system, and the movement is simply a geodesic motion. In the case of fluids, one takes the (formal, for now) continuum limit and considers a fluid configuration to be a sufficiently regular function from M to itself, representing the collection of the positions of each fluid particle, indexed by their initial position. The space $\mathscr{M}$ of these configurations can be thought as an infinite-dimensional manifold, and it should come with notions of kinetic energy and of geodesic movement, describing the movement of a free fluid.

This approach was formalized by Ebin and Marsden in [EM69]. They showed that when the underlying space M is a closed manifold, the configuration set $\mathscr{M}$ of functions $M \to M$ with high enough Sobolev regularity is naturally a Hilbert manifold, and that the kinetic energy defined as the integral of the kinetic energy of every individual particle endows it with a (weak) Riemannian structure. More importantly, if one restricts to the subset $\mathscr{M}_0 \subset \mathscr{M}$ of volume-preserving maps, they prove that this point of view is, as predicted by Arnold, equivalent to the usual Euler equations for inviscid incompressible fluids. This approach is very appealing because it shows that the usual partial differential equations describing those fluids (as is usually written in Eulerian form) are really just an ordinary differential equation, albeit in infinite dimension; this opens the door to many techniques, for instance an application of the Picard–Lindelöf theorem instantly gives local existence and uniqueness when the initial condition is regular enough.

For physical or practical reasons, the purely deterministic nature of the fluid motion equations can be unsatisfactory in view of applications. Uncertainty of measurements or inevitable variations in real experiments naturally invite to consider stochastic versions of fluid dynamics. For instance, computer simulations of fluid evolutions benefit from randomized algorithms for predictive power. Keeping Arnold's heuristics in mind, it is then tempting to consider a perturbation of the equations of fluid motions, in such a way that it can be written as a stochastic differential equation in configuration space rather than as a stochastic partial differential equation, so that it can be studied using classical tools from elementary stochastic analysis. Moreover, from a geometrical point of view, it also seems natural to require the random perturbation to be as intrinsic as possible, as opposed to a perturbation related to an arbitrary random external force imposed on the system.

As we will see below, the stochastic model of kinetic Brownian motion in configuration space that we introduce and study here precisely fulfills the above requirements. For a formal definition of the process in both finite- and infinite-dimensional frameworks, we refer the reader to the original article [ABT15] and to the next Section 2.1. To stay at an introductory and informal level here, let us simply mention that kinetic Brownian motion can be seen as a family of unit velocity stochastic motions in configuration space, indexed by a positive parameter σ. This scalar number adjusts the intensity of noise in the system. As established in [ABT15] in the finite-dimensional framework of Riemannian manifolds, kinetic Brownian motion nicely interpolates between geodesic and Brownian motions. Namely, as the parameter σ goes to zero, the associated sample paths converge to unit speed geodesics, whereas as the σ goes to infinity, when properly rescaled in time, the sample paths behave as standard Brownian motions on the base manifold. The next Figure 1.1 illustrates this homogenization phenomenon for the kinetic Bronwian motion with values in the two dimensional flat torus.

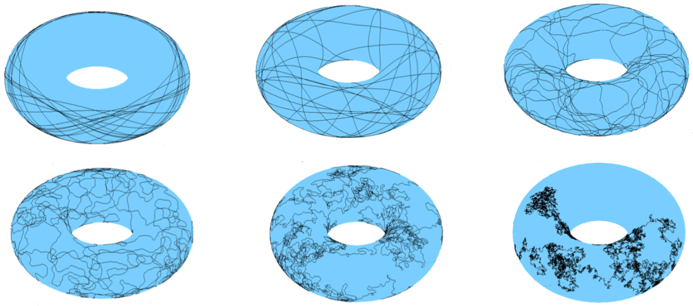

FIGURE 1. Kinetic Brownian motion sample paths on the flat torus for increasing values of the parameter σ.

Going back to Arnold's point on view of fluid mechanics and due to the importance of geodesics in configuration space, it is tempting to ask if kinetic Brownian motion also allows such an interpolation, and in which sense, in the framework of fluid dynamics, namely when the base space is no longer finite-dimensional but replaced by a group of diffeomorphisms or more generally an Hilbert manifold.

Before describing our results, let us stress that the geometric constructions we present below are not particularly tailored to kinetic Brownian motion. A motivated reader may choose any reasonable notion of randomly perturbed line in a Hilbert space, and use our techniques to convert it to a notion of randomly perturbed fluid. The choice of kinetic Brownian motion as our preferred candidate was here motivated by its intrinsic geometric nature and its nice interpolation properties in finite dimension, which in infinite dimension constitutes a typical application of the rough path tools we describe.

1.2. Summary of the results

As announced above, the object of the present work is to properly construct and then study kinetic Brownian motion in the infinite-dimensional framework of the diffeomorphism group $\mathscr{M}$, or volume-preserving diffeomorphism group $\mathscr{M}_0$, of a closed Riemannian manifold M. This includes an interpolation result between the geodesic motion and a Brownian-like flow, i.e. classical fluids and fluids driven purely by noise. Recall that according to Ebin and Marsden, these configuration spaces can be seen as infinite-dimensional Hilbert manifolds. As such and since kinetic Brownian motion is a unit velocity motion, the task is to define and study stochastic processes with values in the unitary tangent bundle of these Hilbert manifolds. Let us stress that the construction of the process in itself is not trivial since the definition of Brownian motion in diffeomophisms groups already raises some difficulties.

Our construction is inspired from the one developed in [ABT15] in the finite-dimensional Riemannian framework: first construct kinetic Brownian motion in the flat tangent space at the desired starting point of the sample path, then use a stochastic development machinery to extend it to the whole manifold. Again, in the infinite-dimensional present context, these two steps are non-trivial, in particular and up to our knowledge, there is no pre-existing notion of such stochastic development on Hilbert manifolds.

The key of our approach, for both the construction of the process and then for the study of its homogenization properties, is the use of rough path techniques. The lift of the fluid equations to the rough paths framework indeed enable us to decompose this exploration into two independent parts:

- On the analytical side, we first construct kinetic Brownian motion in a general infinite-dimensional Hilbert space, see Definition 2.1 below. As in the case of standard Brownian motion in Hilbert space, the definition depends on a bounded symmetric operator on the ambient space. Under some mild conditions on this operator, we then establish its homogenization property, in the continuous and, more importantly for the following, the rough path topology, see Proposition 2.7 and Theorem 2.16 below.

- On the geometric side, we define the Cartan development for infinite-dimensional manifolds endowed with a smooth connection, and we show that for the configuration spaces $\mathscr{M}$ and $\mathscr{M}_0$, it is the solution of a controlled differential equation with smooth coefficients, see Propositions 3.2 and 3.3 below and the subsequent Definitions 3.4 and 3.5.

Then, the fundamental Continuity Theorem of Lyons for solutions of differential equations driven by rough paths acts as a bridge between the worlds of the Hilbert spaces and Hilbert manifolds, and provides a rigorous construction of kinetic Brownian fluids and their Brownian-like limit, as well as a direct proof of the convergence of the first to the latter, as the noise intensity goes to infinity. Precisely, the proper construction of kinetic Brownian motion in configuration space is accomplished via the rigorous Definition 4.1. The large noise limit of the process is finally described in Theorems 4.3 and 4.4 below.

As a result of this rough paths lift approach, and as in the finite-dimensional framework, we thus obtain that kinetic Brownian motion indeed provides an interpolation between the geodesic flow and a Brownian-like flow, as the noise intensity parameter σ ranges from zero to infinity. For $\sigma = 0$, the motion in each diffeomorphism group is geodesic, and it corresponds to the flow of the solutions of Euler's equation in the case of $\mathscr{M}_0$, and to what is sometimes referred to as the EPDiff equation in the case of $\mathscr{M}$. As σ goes to infinity, the process converges to the stochastic development on the manifold of an explicit limit Gaussian process in the Hilbert model space.

The detailed plan of the article is the following. In the next Section 2, after recalling the definition of kinetic Brownian motion in the Euclidean and Riemannian frameworks, we define and study kinetic Brownian motion on a generic infinite-dimensional Hilbert space H. We provide an explicit description of the invariant measure of the velocity process in Section 2.2, and we establish exponential decorrelation identities for the latter in Section 2.3. The invariance principle for the position process associated to the time-rescaled H-valued kinetic Brownian motion is then established in Section 2.4. With the rough paths tools introduced in Section 2.5, Section 2.6 is devoted to the proof of the fact that the canonical rough path above the time-rescaled position process converges weakly as a rough path to the Stratonovich Brownian rough path of a Brownian motion with an explicit covariance. Elements of the geometry of the configuration spaces $\mathscr{M}$ and $\mathscr{M}_0$ are recalled in Section 3. We develop in particular in Section 3.3 and Section 3.4 the material needed to talk about the Cartan development operation as solving an ordinary differential equation driven by smooth vector fields. The final homogenization result, proving the interpolation between geodesic and Brownian flows on the configuration spaces, is proved in Section 4 using the tools of rough paths theory. Appendix A contains the proof of a technical result about the Cartan development in $\mathscr{M}_0$.

1.3. Context and related results

There has been much work recently on random perturbations of Euler's equations, following Holm's seminal article [Hol15]. See [GBH17, CHR18, CFH18, DH18, BdLHLT19] for a sample. In physical terms, this approach adds noise from the *outside*, in the sense that the noise added at a given point y at a given time t is described by some increment $\sigma(t,y)\circ \mathrm{d}W(y)_t$. In contrast, our model considers the influence of the *inside* of the fluid, where the noise at the same point and time is given by some $\mathrm{d}\phi_t \circ \sigma(t,\phi_t^{-1}(y)) \circ \mathrm{d}W(\phi_t^{-1}(y))_t$, where $\phi_t : M \to M$ sends the particle originally at x to its position y at time t. We argue that this might be a good way to tackle lower-scale effects of the fluid, since the unseen randomness (small vortices and eddies) should be transported by the flow, rather than generated by some external force.

From the mathematical point of view, the structure of the noise in the works cited in the previous paragraph is intrinsically linked to the group structure of the diffeomorphism group, and it amounts to disturb Euler's equation for the velocity field by an additive Brownian term, with values in a space of vector fields on the fluid domain M. Our point of view is purely Riemannian, and does not appeal to the group structure of the diffeomorphism group of the fluid domain M. As in the above finite-dimensional setting, we define kinetic Brownian motion on the diffeomorphism group as the Cartan development of its flat counterpart. Unlike the group-oriented point of view, where the running time diffeomorphism is sufficient to describe its infinitesimal increment from the noise, we need here a notion of frame of the tangent space of the running diffeomorphism to build its increment from the noise. We provide nonetheless an explicit description of the invariant measure of the energy of the Eulerian velocity field of kinetic Brownian motion.

In the finite-dimensional case, it was first proved by X.-M. Li in [Li12] that, in the Riemannian context, the time-rescaled position process of kinetic Brownian motion converges weakly to the standard Brownian motion. The base manifold M was assumed to be compact and martingale methods were used to prove that homogenization result. X.-M. Li extended this result in [Li16] to non-compact manifolds subject to a growth condition on their curvature tensor. In [ABT15], Angst, Bailleul and Tardif gave the most general result, assuming only geodesic and stochastic completeness, using as we do here rough paths theory as a work horse to transport a rough path convergence result about kinetic Brownian motion in $\mathbb{R}^d$ to the manifold setting. See also [Li18] for further results in homogeneous spaces, and [Per18] for a generalization of the homogenization result of [ABT15] to anisotropic kinetic Brownian motion, or more general Markov processes on TM. Note that the dynamically obvious convergence of the unrescaled kinetic Brownian motion to the geodesic motion has been studied from the spectral point of view in [Dro17], for compact manifolds with negative curvature, showing that the L^2 spectrum of the generator of the unrescaled kinetic Brownian motion converges to the Pollicott-Ruelle resonances of M. Other examples of homogenization results for Langevin-type processes include works by Hottovy and co-authors, amongst others; see e.g. [BHVW17, HHV16, BW18, LWL19] for quantitative convergence results. See also [Sol95, Kol00, AHK12, Gli11] for other works on Langevin dynamics in a Riemannian manifold.

This kind of homogenization result certainly echoes Bismut's program about his hypoelliptic Laplacian [Bis05, Bis15], whose probabilistic starting point is a similar interpolation result for Langevin process in $\mathbb{R}^d$ and its Cartan development on a Riemannian manifold. The dynamics is lifted to a dynamics on the space of differential forms to take advantage of the correspondence between the cohomology of differential forms and homology of M, via index-type theorems. See [Bis11, Bis15, Bis16, She16] for a sample of the deep results obtained by Bismut and co-authors on the hypoelliptic Laplacian.

Note also that kinetic Brownian motion is the Riemannian analogue of its Lorentzian counterpart, introduced first by Dudley in [Dud66] in Minkowski spacetime in the 60's. See the far reaching related works [FLJ07, Bai10, FLJ11, BF12], on relativistic diffusions in a general Lorentzian setting. No homogenization result is expected for these purely geometric diffusion processes, unless one has an additional non-geometric ingredient, e.g. in the form of a relativistic fluid flow, like in [AF07].

Notations. We gather here a number of notations that are used throughout the article.

- The letter γ stands for a Gaussian measure on a Hilbert space H, with covariance $C_\gamma : H^* \times H^* \to \mathbb{R}$, and associated operator $\overline{C}_\gamma : H \to H$. The scalar product and norm on H are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively.
- We denote by $\mathcal{H}$ the Cameron-Martin space of the measure γ.
- We endow the algebraic tensor space $H \otimes_a H$ with its natural Hilbert norm. This amounts to identify $H \otimes H$ with the space of Hilbert-Schmidt operators on H.
- We use the notation $A \lesssim_p B$ for an inequality of the form $A \leq cB$, with a constant c depending only on p.

2. Kinetic Brownian motion in a Hilbert space

This section is devoted to the construction of kinetic Brownian motion in a general infinite-dimensional Hilbert space. We first briefly recall the definition of the process in the Euclidean and Riemannian settings and then extend it in a natural way to the Hilbertian setting.

2.1. Kinetic Brownian motion in Euclidean and Riemannian settings

This section describes the finite-dimensional situation. Kinetic Brownian motion was originally introduced in [ABT15] in the Riemannian framework as a simple model of stochastic motion with bounded velocity. In its simplest form, that is in the Euclidean space $\mathbb{R}^d$, this process is a hypoelliptic diffusion (x^σ_t, v^σ_t) with values in the state space $T^1\mathbb{R}^d \simeq \mathbb{R}^d \times \mathbb{S}^{d-1}$, solution to the stochastic differential equation

$$\begin{aligned} dx^\sigma_t &= v^\sigma_t \, dt, \\ dv^\sigma_t &= \sigma \, P_{v^\sigma_t}(\circ \, dW_t), \end{aligned}$$

for $P_a : \mathbb{R}^d \to \langle a \rangle^\perp$ the orthogonal projection on the orthogonal of $\langle a \rangle$, for $a \neq 0$ in $\mathbb{R}^d$, and W a standard $\mathbb{R}^d$-valued Brownian motion. The non-negative scalar parameter σ adjusts the intensity of noise in the model. The generator of the diffusion is some extension of the operator

$$v \cdot \nabla_x + \frac{\sigma^2}{2} \Delta_v$$

acting on smooth functions with compact support, where ∇_x is the gradient with respect to the position $x \in \mathbb{R}^d$ and Δ_v is the spherical Laplacian acting on the velocity component $v \in \mathbb{S}^{d-1}$.

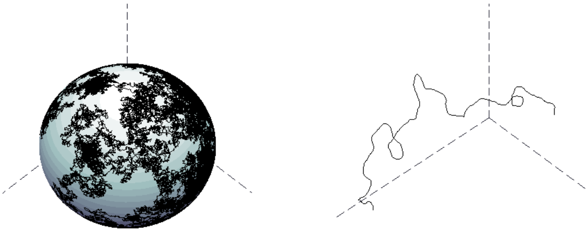

Figure 2. A sample of kinetic Brownian motion in $\mathbb{R}^3$, where the velocity appears to the left and the position to the right

Over a Riemannian manifold M, we can define kinetic Brownian motion up to explosion as a process $(z^\sigma, \dot z^\sigma)$ in the unit tangent bundle T^1M with the same generator, mutatis mutandis. It turns out to be equivalent to define it as the Cartan development of its Euclidean counterpart, in the following sense. Denoting by $/\!/^z_t : T_{z_0}M \to T_{z_t}M$ the parallel transport over a C^1 curve z over M, and given another C^1 curve x with values in the flat space $T_{z_0}M$, we say that z is the Cartan development of x if the process

$$t \mapsto (/\!/^z_t)^{-1}(\dot z_t) \in T_{z_0}M$$

of the velocity read in the fixed space $T_{z_0}M$ coincides with the velocity of x. In our case, a process $(z^\sigma, \dot z^\sigma)$ is a kinetic Brownian motion over M with parameter σ, as defined using the generator characterization, if and only if the velocity $\dot z^\sigma$, up to parallel transport, is a spherical Brownian motion on the unit sphere of $T_{z_0}M$ run at speed σ^2.

The Cartan development is often described as the solution of a controlled differential equation in some bigger space OM known as the orthonormal frame bundle. It consists of pairs (z,e) with $z \in M$ and e an isometry from $\mathbb{R}^d$ to T_zM, and one can prove that z is the Cartan development of x if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}(z_t,e_t) = \big(H(z_t,e_t)\circ e_0^{-1}\big)\left(\frac{\mathrm{d}x_t}{\mathrm{d}t}\right),$$

where $H(z,e) : \mathbb{R}^d \to T_{(z,e)}OM$ is the so-called horizontal lift, a linear operator depending smoothly on (z,e). This is typically the type of equations considered in rough path theory. In [ABT15], Angst, Bailleul and Tardif proved not only that the position component x^σ of Euclidean kinetic Brownian motion, rescaled as $t \mapsto x^\sigma_{\sigma^2 t}$, converges in distribution to a constant multiple of Brownian motion in $C^0([0,1],\mathbb{R}^d)$, but also that its canonical rough path lift converges in distribution to the corresponding Brownian rough path in the rough path topology. Accordingly, the continuity theorem of Lyons shows that the manifold-valued kinetic Brownian motion $t \mapsto z^\sigma_{\sigma^2 t}$ converges to a Brownian motion in M runs a some deterministic speed. Our objective in this section is to generalize the rough path convergence from the Euclidean setting to its Hilbert space analogue.

2.2. Kinetic Brownian motion in a Hilbert space

We now extend the above construction of kinetic Brownian motion to the Hilbertian framework. We first recall basic results on Brownian motion in H, and refer the reader to the nice lecture notes [Hai12, Str93] for short and detailed accounts.

We fix a Hilbert space H. Recall that a **Gaussian probability measure on** H is a Borel measure γ such that $\ell^*\gamma$ is a real Gaussian probability on $\mathbb{R}$, for every continuous linear functional $\ell : H \to \mathbb{R}$. Fernique's theorem [Fer70] ensures that

$$\int_H \exp\left(a\|x\|^2\right)\gamma(dx) < \infty,$$

for a small enough positive constant a. It follows that the covariance

$$C_\gamma(\ell,\ell') := \int \ell(x)\ell'(x)\,\gamma(dx), \quad \ell,\ell' \in H^*$$

is a well-defined continuous bilinear operator on $H^* \times H^*$. Denote by $\iota : H \to H^*$ the canonical isomorphism provided by the scalar product. One can then define a continuous symmetric operator $\overline{C}_\gamma : H \to H$, by the identity

$$\big(\overline{C}_\gamma(h),k\big) = C_\gamma(\iota h,\iota k),$$

for all $h,k \in H$. It has finite trace equal to

$$\operatorname{tr}(\overline{C}_\gamma) = \int \|x\|^2\,\gamma(dx).$$

Conversely, one can associate to any trace-class symmetric operator $\overline{C} : H \to H$, a Gaussian measure γ on H whose covariance $C_\gamma(\ell,\ell) = \overline{C}(\iota^{-1}\ell,\iota^{-1}\ell)$, for all $\ell \in H$. Since $\overline{C}_\gamma$ is compact and non-negative, there exists an orthonormal basis (e_n) of H, such that

$$\overline{C}_\gamma(e_n) = \alpha_n^2 e_n,$$

for non-negative and non-increasing eigenvalues α_n with $\sum \alpha_n^2 < \infty$. We define a Hilbert space $\mathcal{H}$ by choosing $\big(\alpha_n e_n\big)$ as an orthonormal basis for it. The space $\mathcal{H}$ is continuously embedded inside H. Let (X^n) stand for a sequence of independent, identically distributed, real-valued Gaussian random variables with zero mean and unit variance, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the series

$$\sum_n X^n \alpha_n e_n$$

converges in $L^2(\Omega, H)$, and has distribution γ.

Fix a positive time horizon $T \in (0, \infty]$. An **$\mathcal{H}$-Brownian motion in H**, on the time interval $[0,T)$ is a random H-valued continuous path W on $[0,T)$, with stationary, independent increments such that the distribution of W_1 is a Gaussian probability measure γ on H. A simple construction is provided by taking a sequence (W_t^n) of independent, identically distributed, real-valued standard Brownian motions, and setting

$$W_t := \sum_n W_t^n \alpha_n e_n.$$

Denote by S the unit sphere of H, and let

$$P_a : H \to H$$

stand for the orthogonal projection on $\langle a\rangle^\perp$, for $a \neq 0$.

We can now extend the definition of kinetic Brownian motion to the Hilbertian framework, by mimicking the Euclidean one. As above, the speed parameter σ below is a non-negative real number.

Definition 2.1. *The* **$\mathcal{H}$-spherical Brownian motion** v_t^σ *on S is defined as the (infinite lifetime) solution to the Stratonovich stochastic differential equation*

$$dv_t^\sigma = \sigma\, P_{v_t^\sigma}(\circ dW_t) \tag{1}$$

associated to a given initial condition $v_0^\sigma \in S$. Then, the associated **kinetic Brownian motion** *in H is defined as the (infinite lifetime) solution (x_t^σ, v_t^σ) in T^1H of the system*

$$\begin{aligned} dx_t^\sigma &= v_t^\sigma\, dt, \\ dv_t^\sigma &= \sigma\, P_{v_t^\sigma}(\,\circ\, dW_t), \end{aligned}$$

starting from a given initial condition $(x_0^\sigma, v_0^\sigma) \in T^1H$.

Remark 2.2. *As there is no notion of canonical Brownian motion in a general Hilbert space, there is thus no canonical notion of kinetic Brownian motion. Both processes do depend on the embedding of $\mathcal{H}$ in H explicited above.*

Having properly defined kinetic Brownian motion in a general Hilbert space, we now concentrate on its homogenization properties as the noise intensity σ goes to infinity. We prove in the next Section 2.3 that the velocity process (v_t^σ) converges exponentially fast in Wasserstein distance to an explicit invariant probability measure, for any initial velocity v_0. An invariance principle in the continuous topology for the time-rescaled position process $(x^\sigma_{\sigma^2 t})$ is then obtained as a consequence in Section 2.4.

We recall in Section 2.5 the material we need from rough paths theory in this work, and prove in Section 2.6 that the canonical rough path associated to the time-rescaled process $(x^\sigma_{\sigma^2 t})$ converges weakly as a rough path to an explicit Stratonovich Brownian rough path. This rough path lift of the above mentioned invariance principle will in fact be crucial to easily deduce its analogue in the general framework of Hilbert manifolds, and in particular when dealing with diffeomorphism groups in Sections 4.1 and 4.2.

2.3. Exponential mixing of the velocity process

We study here the ergodic properties of the $\mathcal{H}$-spherical Brownian motion (v_t^σ) solution of Equation (1), i.e. the velocity component of kinetic Brownian motion. To simplify the expressions in the sequel, let us denote by Z the finite normalization constant $\int_H \frac{1}{\|u\|}\gamma(du)$.

Theorem 2.3. *The image under the projection $u \mapsto u/\|u\|$ of the measure $\frac{1}{Z}\frac{1}{\|u\|}\gamma(du)$ in the ambient space H is a probability measure μ on S that is invariant for the dynamics of (v_t^σ), for any positive speed parameter σ.*

This statement generalizes Proposition 1.1 of [Per18] to the present infinite-dimensional setting. The above description of the invariant measure μ as an image measure under the projection map actually coincides with the finite-dimensional description given in the latter reference.

Proof. When written in Itô form, the stochastic differential equation (1) defining the process $(v_t^\sigma)_{t\geq 0}$ reads

$$dv_t^\sigma = -\frac{\sigma^2}{2}\Big(\operatorname{tr}(\overline{C}_\gamma)v_t^\sigma + \overline{C}_\gamma(v_t^\sigma) - 2C_\gamma(v_t, v_t)v_t^\sigma\Big)dt + \sigma\, P_{v_t^\sigma}(dW_t), \tag{2}$$

and setting $v_t^{\sigma,i} := \langle v_t^\sigma, e_i\rangle$, for any integer i, we have

$$\begin{aligned} dv_t^{\sigma,i} &= -\frac{\sigma^2}{2}\left[\sum_n \alpha_n^2 + \alpha_i^2 - 2\sum_n \alpha_n^2 |v_t^{\sigma,n}|^2\right] v_t^{\sigma,i}\, dt \\ &\quad + \sigma\left[\alpha_i dW_t^i - v_t^{\sigma,i}\sum_n \alpha_n v_t^{\sigma,n} dW_t^n\right]. \end{aligned}$$

As in the finite-dimensional anisotropic case treated in [Per18], it is actually easier to work with an H-valued lift of this S-valued process. We introduce for that purpose the process $(u_t^\sigma)_{t\geq 0}$ solution of the Stratonovich stochastic differential equation

$$du_t^\sigma = -\frac{\sigma^2}{2}\|u_t^\sigma\|^2 u_t^\sigma dt + \sigma\|u_t^\sigma\|{\circ}dW_t;$$

equivalently, in Itô form and coordinate-wise, setting $u_t^{\sigma,i} := \langle u_t^\sigma, e_i\rangle$ as above, we have

$$du_t^{\sigma,i} = \frac{\sigma^2}{2}\left(-\|u_t^\sigma\|^2 + \alpha_i^2\right) u_t^{\sigma,i} dt + \sigma\|u_t^\sigma\|\alpha_i dW_t^i.$$

A direct application of Itô's formula then shows that $u_t^{\sigma,i}/\|u_t^\sigma\|$ satisfies the same stochastic differential equation as $v_t^{\sigma,i}$, for all i, so the two S-valued processes $(v_t^\sigma)_{t\geq 0}$ and $(u_t^\sigma/\|u_t^\sigma\|)_{t\geq 0}$ have the same distributions. As in the finite-dimensional case, one can then check by a direct computation that the measure $\|u\|^{-1}\gamma(du)$ on H is invariant for the processes (u_t^σ); this implies the statement of Theorem 2.3.

Alternatively, one can bypass computations and argue using Malliavin calculus as follows. Denote by L the infinitesimal generator of the process (u_t^σ). Set $V(u) := u/\|u\|^2$ for $u \neq 0$, and let Δ_γ denote the Laplace operator associated with the covariance C_γ with weights (α_n^2),. We then have for any test function f and any $u \in H$

$$(Lf)(u) = \frac{\sigma^2}{2}\|u\|^2 (L_0 f)(u),$$

with

$$(L_0 f)(u) := \Delta_\gamma f(u) - u\nabla f(u) + C_\gamma\big(V(u), \nabla f(u)\big).$$

One then has for any test function f, with usual notations D for the gradient and δ for the divergence operator associated with the Gaussian measure γ,

$$\begin{aligned}\int_H (Lf)(u)\|u\|^{-1}\gamma(du) &= \frac{\sigma^2}{2}\int_H (L_0 f)(u)\|u\|\gamma(du)\\ &= \sigma^2\,\mathbb{E}\Big[\big(-\delta DF + \langle V, DF\rangle_{C_\gamma}\big)\,\|W\|\Big]\\ &= \mathbb{E}\Big[\big(-\delta \underbrace{D\|W\|}_{=\frac{W}{\|W\|}} + \delta \underbrace{V\|W\|}_{=\frac{W}{\|W\|}}\big)F\Big] = 0.\end{aligned}$$

□

We consider in this section the mixing properties of the spherical process $(v_t^\sigma)_{t\geq 0}$ with unit speed parameter $\sigma = 1$. To simplify the expressions, we drop momentarily the exponents σ from all our notations. Our objective is to show that the spherical process

$$(v_t)_{t\geq 0} := (v_t^1)_{t\geq 0}$$

is exponentially mixing. Recall that the 2 and 1-Wasserstein distances are defined for any probability measures μ, ν on S by the identities

$$\begin{aligned}\mathcal{W}_2(\lambda,\nu) &= \inf\Big\{\mathbb{E}\big[\|X-Y\|^2\big]; X\sim\lambda, Y\sim\nu\Big\},\\ \mathcal{W}_1(\lambda,\nu) &= \inf\Big\{\mathbb{E}\big[\|X-Y\|\big]; X\sim\lambda, Y\sim\nu\Big\}\\ &= \sup\left\{\int f\,d(\lambda-\nu); |f|_{\mathrm{Lip}}\leq 1\right\},\end{aligned}$$

where the infimum is taken over all couplings $\mathbb{P}$ of $X\sim\lambda$ and $Y\sim\nu$, and the supremum over all 1-Lipscthiz functions $f : S\to\mathbb{R}$. The first two equalities are definitions, the last one is the Kantorovich-Rubinstein duality principle. Note that $\mathcal{W}_1\leq\mathcal{W}_2$.

Proposition 2.4. *Assume that*

$$3\alpha_0^2 < \mathrm{tr}(\overline{C}_\gamma). \tag{3}$$

There exists a positive time τ such that for any probability measures λ and ν on the unit sphere S of H, we have

$$\mathcal{W}_2(P_t^*\lambda, P_t^*\nu)\leq e^{-t/\tau}\mathcal{W}_2(\lambda,\nu),$$

for all $t\geq 0$. In particular, the invariant measure μ is unique, and for any probability measure λ on the sphere S, and $t\geq 0$, we have

$$\mathcal{W}_2(P_t^*\lambda,\mu)\leq 2e^{-t/\tau}. \tag{4}$$

The role of the trace condition (3) will be clear from the proof. If we have the freedom to choose the covariance C_γ of the Brownian noise, this is not a constraint. Note that the rougher the noise, that is the more slowly the sequence of the eigenvalues α_n converges to 0, the easier it is to satisfy condition (3). We shall see in Section 4 that it holds automatically in a number of relevant examples of random dynamics in the configuration space of a fluid flow.

Proof. Denote by $\mathbb{P}$ the law of the Brownian motion (B_t) with covariance C_γ, and by $\mathbb{P}_v$ the law of the solution of Equation (1) with $\sigma=1$, starting from $v\in S$. Denote by $\mathbb{E}$ and $\mathbb{E}_v$ the associated expectations operators. Recall that the notation (a,b) stands for the scalar product of a and b in H. Fix $v_0, w_0\in S$, and consider the two diffusion processes (v_t) and (w_t), started from v_0 and w_0, respectively, and solutions of the Itô stochastic differential equations

$$\begin{aligned}dv_t &= -\frac{1}{2}\Big(\mathrm{tr}(\overline{C}_\gamma)v_t + \overline{C}_\gamma(v_t) - 2C_\gamma(v_t,v_t)v_t\Big)dt + P_{v_t}(dW_t),\\ dw_t &= -\frac{1}{2}\Big(\mathrm{tr}(\overline{C}_\gamma)w_t + \overline{C}_\gamma(w_t) - 2C_\gamma(w_t,w_t)w_t\Big)dt + P_{w_t}(dW_t).\end{aligned}$$

Comparing with Equation (2), it is clear that (v_t) has law $\mathbb{P}_{v_0}$ and (w_t) has law $\mathbb{P}_{w_0}$. Moreover, Itô's formula yields

$$\begin{aligned} d(v_t, w_t) = \Big(\operatorname{tr}(\overline{C}_\gamma) - C_\gamma(v_t, v_t) - C_\gamma(w_t, w_t) - C_\gamma(v_t, w_t)\Big)\big(1 - (v_t, w_t)\big)dt \\ + \big(1 - (v_t, w_t)\big)\Big((v_t, dW_t) + (w_t, dW_t)\Big), \end{aligned}$$

or equivalently, setting

$$N_t := \frac{1}{2}\|w_t - v_t\|^2 = 1 - (v_t, w_t),$$

we get

$$\begin{aligned} dN_t = -\Big(\operatorname{tr}(\overline{C}_\gamma) - C_\gamma(v_t, v_t) - C_\gamma(w_t, w_t) - C_\gamma(v_t, w_t)\Big)N_t dt \\ - N_t\big((v_t, dW_t) + (w_t, dW_t)\big). \end{aligned} \tag{5}$$

Now remark that since the sequence (α_n) is non-increasing, we have

$$C_\gamma(v, v) = \sum_{n \geq 0} \alpha_n^2 |v_n|^2 \leq \alpha_0^2,$$

for any $v \in S$. Taking the expectation under $\mathbb{P}$ in equation (5), we have from Grönwall inequality

$$\mathbb{E}[N_t] \leq e^{-t(\operatorname{tr}(C_\gamma) - 3\alpha_0^2)}\mathbb{E}[N_0],$$

that is

$$\mathbb{E}[\|v_t - w_t\|^2] \leq e^{-t(\operatorname{tr}(C_\gamma) - 3\alpha_0^2)}\, \|x - y\|^2.$$

The conclusion of the statement follows. ☐

Remark that $\mathbb{E}_\mu[v_t] = 0$, as a consequence of the symmetry properties of the invariant measure μ.

Corollary 2.5. *For any $v_0 \in S$, we have*

$$\big\|\mathbb{E}_{v_0}[v_t]\big\| \leq 2e^{-t/\tau}.$$

The process (v_t) is stationary if v_0 has distribution μ; it can then be extended into a two sided process defined for all real times. Denote by $(\mathcal{F}_t)_{t\in\mathbb{R}}$ the complete filtration generated by (v_t) on the probability space where it is defined. Set $\mathcal{F}_{\leq 0} := \sigma\big(\mathcal{F}_t\,;\, t \leq 0\big)$ and $\mathcal{F}_{\geq s} := \sigma\big(\mathcal{F}_t\,;\, t \geq s\big)$, for any real time s. Recall that the mixing coefficient $\alpha(s)$ of the velocity process v is defined, for $s > 0$, by the formula

$$\alpha(s) := \sup_{A\in\mathcal{F}_{\leq 0}, B\in\mathcal{F}_{\geq s}} \big|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\big|.$$

The following fact will be useful to get for free the independence of the increments of the limit processes obtained after proper rescalings of functionals of (v_t).

Corollary 2.6. *The mixing coefficient $\alpha(s)$ tends to 0 as s increases to ∞.*

Proof. As a preliminary remark, recall the definition of the lift (u_t^σ) to H of (v_t^σ), introduced in the proof of Theorem 2.3. This process is strong Feller, as it can be seen to satisfy a Bismut-Li integration by parts formula. See e.g. Peszat and Zabczyk' seminal paper [PZ95], and Wang and Zhang's extension [WZ10] to unbounded drift and diffusivity. The velocity process (v_t^σ) is thus itself a strong Feller diffusion, and if one denotes by (P_t) its transition semigroup, the functions $P_1 g$, for g measurable, bounded by 1, are all Lipschitz continuous, with a finite common upper bound L for their Lipschitz constants.

Now, it follows from the Markovian character of the dynamics of (v_t), and the Feller property of its semigroup, that it suffices to see that

$$\mathbb{E}\big[f(v_0)g(v_s)\big] \tag{6}$$

tends to 0 as s goes to ∞, for any real-valued continuous functions f, g on the unit sphere S, with null mean with respect to the invariant measure μ, uniformly with respect to f and g with L^∞-norm 1. Writing further

$$\mathbb{E}\Big[f(v_0)\mathbb{E}\big[g(v_s)|v_{s_1}\big]\Big] = \mathbb{E}\Big[f(v_0)\,(P_1 g)(v_{s-1})\Big],$$

for $s > 1$, and using the strong Feller property of the semigroup of the diffusion process (v_t), we can further assume that the function g in (6) is $L\|g\|_\infty$-Lipschitz continuous. Let w_g stand for its uniform modulus of continuity. For each s, denote by $(v_s, \overline{v}_s)$ a $\mathcal{W}_1$-optimal coupling of the measures $P_s^*\delta_{v_0}$ and μ, for a deterministic v_0, so we have

$$\mathbb{E}\big[|v_s - \overline{v}_s|\big] = \mathcal{W}_1\big(P_s^*\delta_{v_0}, \mu\big).$$

Using the fact that $\int g d\mu = 0$, one then has

$$\begin{aligned}\big|\mathbb{E}\big[f(v_0)g(v_s)\big]\big| &= \Big|\mathbb{E}\Big[f(v_0)\,\mathbb{E}\big[g(v_s)|v_0\big]\Big]\Big| \\ &\leq \|f\|_\infty\,\mathbb{E}\Big[w_g\big(|v_s - \overline{v}_s|\big)\Big] \\ &\leq L\|f\|_\infty\|g\|_\infty\,\mathbb{E}\big[|v_s - \overline{v}_s|\big],\end{aligned}$$

so the statement follows from Proposition 2.4. □

2.4. Invariance principle for the position process

We assume in all of this section that the initial condition v_0 of the velocity process of kinetic Brownian motion is distributed according to its invariant probability measure μ, described in Theorem 2.3.

Pick $1/3 < \alpha \leq 1/2$. We prove in this section that the distribution in $C^\alpha([0,1], H)$ of the time-rescaled position process $(x_{\sigma^2 t}^\sigma)$ of kinetic Brownian motion in T^1H converges to the distribution of a Brownian motion in H with an explicit covariance, given in identity (7) of Proposition 2.7 below. The usual invariance principles in Hilbert spaces consider weak convergence in $C^0([0,1], H)$, so we need an extra tightness estimate provided in Section

2.4.1 to complete the program. To make the most out of the convergence results from Section 2.3, set

$$X_t^\sigma := x_{\sigma^2 t}^\sigma;$$

we have

$$X_t^\sigma - X_s^\sigma = \int_{\sigma^2 s}^{\sigma^2 t} v_{\sigma^2 u} du = \frac{1}{\sigma^2} \int_{\sigma^4 s}^{\sigma^4 t} v_u du,$$

with $(v_t) = (v_t^1)$, the spherical Brownian motion run at speed $\sigma^2 = 1$.

Proposition 2.7. *For every* $0 < \alpha < 1/2$*, the distribution in* $\mathcal{C}^\alpha([0,1], H)$ *of the process* (X_t^σ) *converges as* σ *goes to* ∞ *to the Brownian motion on* H *with covariance operator*

$$C(\ell, \ell') := \int_0^\infty \mathbb{E}\Big[\ell(v_0)\ell'(v_t) + \ell'(v_0)\ell(v_t)\Big]\, dt, \tag{7}$$

for $\ell, \ell' \in H^*$*.*

2.4.1. Tightness in Hölder spaces. We dedicate this section to proving the following uniform estimate.

Proposition 2.8. *For any* $p \geq 2$*, we have*

$$\sup_{\sigma > 0} \mathbb{E}\big[\|X_t^\sigma - X_s^\sigma\|^p\big] \lesssim_p |t-s|^{p/2}.$$

It follows from Kolmogorov-Lamperti tightness criterion that the laws of X^σ form a tight family in $\mathcal{C}^\alpha([0,1], H)$, for any $0 < \alpha < 1/2$. Note that for $T = \sigma^4(t-s) > 0$, we have

$$\big\|X_t^\sigma - X_s^\sigma\big\| \overset{\mathcal{L}}{=} \frac{1}{\sigma^2} \left\| \int_0^{\sigma^4(t-s)} v_u\, du \right\| = |t-s| \cdot \frac{1}{\sqrt{T}} \left\| \int_0^T v_u\, du \right\|,$$

so Proposition 2.8 is a consequence of the estimate

$$\mathbb{E}\left[\left\| \int_0^T v_t\, dt \right\|^p\right] \lesssim_p T^{p/2}.$$

We translate our problem in discrete time, writing

$$\int_0^T = \sum_{k<T} \int_k^{k+1}$$

to work with the correlations between different integral slices, and compare this sequence to martingale differences. There is an abundant literature on the subject; we follow here the approach of C. Cuny [Cun17].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n)_{n \geq n_0}$, where $-\infty \leq n_0 \leq 0$, and let $(X_n)_{n \geq n_0}$ be H-valued random variables such that each X_n is measurable with respect to $\mathcal{F}_n$. Recall that $(X_n)_{n \geq 0}$ is said to be a **martingale difference** with respect to $(\mathcal{F}_n)$ if each X_n is integrable and $\mathbb{E}\big[X_{n+1} | \mathcal{F}_n\big] = 0$, for all $n \geq n_0$. The following result is an elementary consequence of the Burkholder-Davis-Gundy and Jensen inequalities.

Lemma 2.9. *Let X be an H-valued martingale difference with moments of order $p \geq 2$. Then*

$$\mathbb{E}\big[|X_0 + \cdots + X_{n-1}|^p\big]^{\frac{1}{p}} \lesssim_p \sqrt{n}\,\left(\frac{1}{n}\Big(\mathbb{E}\big[|X_0|^p\big] + \cdots + \mathbb{E}\big[|X_{n-1}|^p\big]\Big)\right)^{\frac{1}{p}}.$$

In particular, if X is stationary, then

$$\mathbb{E}\Big[|X_0 + \cdots + X_{n-1}|^p\Big]^{\frac{1}{p}} \lesssim_p \sqrt{n}\,\|X_0\|_{L^p}.$$

Assume from now on that we are given a sequence $(X_n)_{\geq n_0}$ of integrable H-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. For $j \in \mathbb{Z}$, and $k \geq 0$, define the σ-algebra

$$\mathcal{F}_j^{(k)} := \mathcal{F}_{j2^k},$$

and set

$$Y_j^{(k)} := \mathbb{E}\Big[X_{j2^k} + \cdots + X_{j2^k+(2^k-1)}\Big|\mathcal{F}_{j-1}^{(k)}\Big].$$

(It may not make sense for all j, k, depending on how far in the past the σ-algebras $(\mathcal{F}_n)$ are defined.) Note that

$$Y_j^{(\ell+1)} = \mathbb{E}\Big[Y_{2j}^{(\ell)} + Y_{2j+1}^{(\ell)}\Big|\mathcal{F}_{j-1}^{(\ell+1)}\Big],$$

so

$$M_j^{(\ell)} := Y_{2j}^{(\ell)} + Y_{2j+1}^{(\ell)} - Y_j^{(\ell+1)}$$

is a stationary martingale difference with respect to the filtration $\big(\mathcal{F}_j^{(\ell+1)}\big)_{j\geq 0}$. We use the classical martingale/co-boundary decomposition to prove the next result.

Lemma 2.10. *Fix $p \geq 2$, and assume that $\mathcal{F}_n$ is defined for $n \geq -2^{k+1}$, then*

$$\mathbb{E}\Big[\big|Y_0^{(0)} + \cdots + Y_{2^k-1}^{(0)}\big|^p\Big]^{\frac{1}{p}} \lesssim_p \sum_{0\leq j\leq k} 2^{(k-j)/2}\left(\frac{1}{2^{k-j}}\Big(\mathbb{E}\big[|Y_0^{(j)}|^p\big] + \cdots + \mathbb{E}\big[|Y_{2^{k-j}-1}^{(j)}|^p\big]\Big)\right)^{\frac{1}{p}}.$$

In particular, if the sequence (X_n) is stationary, then

$$\mathbb{E}\big[\big|Y_0^{(0)} + \cdots + Y_{2^k-1}^{(0)}\big|^p\big]^{\frac{1}{p}} \lesssim_p 2^{k/2}\Big(\mathbb{E}\big[|Y_0^{(0)}|^p\big]^{\frac{1}{p}} + \cdots + 2^{-k/2}\mathbb{E}\big[|Y_0^{(k)}|^p\big]^{\frac{1}{p}}\Big).$$

Proof. For any $0 \leq j \leq k$, set $n_j := 2^{k-j}$; note that $n_k = 1$. We have for $j < k$ the identity

$$\begin{aligned} Y_0^{(j)} + \cdots + Y_{n_j-1}^{(j)} &= \big(Y_0^{(j)} + Y_1^{(j)}\big) + \cdots + \big(Y_{2n_{j+1}-2}^{(j)} + Y_{2n_{j+1}-1}^{(0)}\big) \\ &= M_0^{(j)} + \cdots + M_{n_{j+1}-1}^{(j)} + Y_0^{(j+1)} + \cdots + Y_{n_{j+1}-1}^{(j+1)}. \end{aligned}$$

By induction we get

$$Y_0^{(0)} + \cdots + Y_{n-1}^{(0)} = \big(M_0^{(0)} + \cdots + M_{n_1-1}^{(0)}\big) + \cdots + \big(M_0^{(k-1)}\big) + Y_0^{(k)}.$$

Because $M^{(j)}$ is a martingale difference, we know from Lemma 2.9 that

$$\begin{aligned}\mathbb{E}\big[\big|M_0^{(j)}+\cdots+M_{n_{j+1}-1}^{(j)}\big|^p\big]^{\frac{1}{p}}\\ &\lesssim_p \sqrt{n_{j+1}}\cdot\bigg(\frac{1}{n_{j+1}}\Big(\mathbb{E}\big[\big|M_0^{(j)}\big|^p\big]+\cdots+\mathbb{E}\big[\big|M_{n_{j+1}}^{(j)}\big|^p\big]\Big)\bigg)^{\frac{1}{p}}.\end{aligned}$$

We also know that

$$M_{2^k}^{(j)} = Y_{2^{k+1}}^{(j)} + Y_{2^{k+1}+1}^{(j)} - \mathbb{E}\big[Y_{2^{k+1}}^{(j)} + Y_{2^{k+1}+1}^{(j)}\big|\mathcal{F}_{2^k-1}^{(j+1)}\big],$$

so we have

$$\begin{aligned}\mathbb{E}\big[\big|M_{2^k}^{(j)}\big|^p\big]^{\frac{1}{p}} &\leq \mathbb{E}\big[\big|Y_{2^{k+1}}^{(j)}\big|^p\big]^{\frac{1}{p}} + \mathbb{E}\big[\big|Y_{2^{k+1}+1}^{(j)}\big|^p\big]^{\frac{1}{p}} + \mathbb{E}\Big[\mathbb{E}\big[|Y_{2^{k+1}}^{(j)}|^p\big|\mathcal{F}_{-1}^{(j+1)}\big]\Big]^{\frac{1}{p}}\\ &\quad + \mathbb{E}\Big[\mathbb{E}\big[|Y_{2^{k+1}+1}^{(j)}|^p\big|\mathcal{F}_{-1}^{(j+1)}\big]\Big]^{\frac{1}{p}}\\ &\leq 2\mathbb{E}\big[\big|Y_{2^{k+1}}^{(j)}\big|^p\big]^{\frac{1}{p}} + 2\mathbb{E}\big[\big|Y_{2^{k+1}+1}^{(j)}\big|^p\big]^{\frac{1}{p}}.\end{aligned}$$

Putting it all together, we obtain

$$\begin{aligned}\mathbb{E}\big[\big|Y_0^{(0)}+\cdots+Y_{2^k-1}^{(0)}\big|^p\big]^{\frac{1}{p}}\\ &\lesssim_p \sum_{0\leq j<k}\sqrt{n_{j+1}}\cdot\bigg(\frac{1}{2n_{j+1}}\Big(\mathbb{E}\big[\big|Y_0^{(j)}\big|^p\big]+\cdots+\mathbb{E}\big[\big|Y_{2n_{j+1}}^{(j)}\big|^p\big]\Big)\bigg)^{\frac{1}{p}}\\ &\quad + \mathbb{E}\big[\big|Y_0^{(k)}\big|^p\big]^{\frac{1}{p}}\\ &\lesssim_p \sum_{0\leq j\leq k} 2^{(k-j)/2}\cdot\bigg(\frac{1}{2^{k-j}}\Big(\mathbb{E}\big[\big|Y_0^{(j)}\big|^p\big]+\cdots+\mathbb{E}\big[\big|Y_{2^{k-j}-1}^{(j)}\big|^p\big]\Big)\bigg)^{\frac{1}{p}}.\end{aligned}$$

□

Proof of Proposition 2.8. It is enough to prove that we have for any $T \geq 1$ and $p \geq 2$, the estimate

$$\mathbb{E}\left[\left|\int_0^T v_t\,dt\right|^p\right] \lesssim_p T^{p/2}.$$

Fix the integer k such that $T/2 \leq 2^k < T$, and define

$$X_j := \int_{jT2^{-k}}^{(j+1)T2^{-k}} v_t\,dt, \qquad \mathcal{F}_j = \sigma\Big(v_s, s\leq (j+1)T2^{-k}\Big).$$

Since we assume that v_0 is distributed according to an invariant probability measure, we can actually have our process started for a time arbitrarily far in the past, so we can assume that $\mathcal{F}_j$ is well-defined for any $j \geq -2^{k+1}$. We can then write

$$\begin{aligned}\int_0^T v_t\,dt &= \big(X_0 - \mathbb{E}[X_0|\mathcal{F}_{-1}]\big) + \cdots + \big(X_{2^k-1} - \mathbb{E}[X_{2^k-1}|\mathcal{F}_{2^k-2}]\big)\\ &\quad + \mathbb{E}\big[X_0|\mathcal{F}_{-1}\big] + \cdots + \mathbb{E}\big[X_{2^k-1}|\mathcal{F}_{2^k-2}\big].\end{aligned}$$

The first sum is a stationary martingale difference with respect to the σ-algebra $(\mathcal{F}_j)_{j\geq 0}$; the second is the subject of the previous lemma. One then has the estimate

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^T v_t\,dt\right|^p\right]^{\frac{1}{p}} &\lesssim_p 2^{k/2}\,\mathbb{E}\Big[\big|X_0 - \mathbb{E}[X_0|\mathcal{F}_{-1}]\big|^p\Big]^{\frac{1}{p}} \\ &\quad + 2^{k/2}\left(\mathbb{E}\Big[\big|Y_0^{(0)}\big|^p\Big]^{\frac{1}{p}} + \cdots + 2^{-k/2}\,\mathbb{E}\Big[\big|Y_0^{(k)}\big|^p\Big]^{\frac{1}{p}}\right) \\ &\lesssim_p \sqrt{T}\left(\mathbb{E}\Big[\big|X_0\big|^p\Big]^{\frac{1}{p}} + \mathbb{E}\Big[\big|Y_0^{(0)}\big|^p\Big]^{\frac{1}{p}} + \cdots + 2^{-k/2}\,\mathbb{E}\Big[\big|Y_0^{(k)}\big|^p\Big]^{\frac{1}{p}}\right)\end{aligned}$$

with the notations of Lemma 2.10. In our setting,

$$\|X_0\|_{L^p} = \mathbb{E}\left[\left|\int_0^{T2^{-k}} v_t\,dt\right|^p\right]^{\frac{1}{p}} \leq \left(T2^{-k}\right)^{\frac{p}{p}} \leq 2$$

and

$$Y_0^{(j)} = \mathbb{E}\left[\int_{2^jT2^{-k}}^{2^{j+1}T2^{-k}} v_t\,dt\,\middle|\,\mathcal{F}_{-1}\right] = \mathbb{E}_{v_0}\left[\int_{2^jT2^{-k}}^{2^{j+1}T2^{-k}} v_t\,dt\right].$$

Note that we have from Corollary 2.5

$$\begin{aligned}\left|\mathbb{E}_{v_0}\left[\int_{2^jT2^{-k}}^{2^{j+1}T2^{-k}} v_t\,dt\right]\right| \leq \int_{2^jT2^{-k}}^{2^{j+1}T2^{-k}} \big|\mathbb{E}_{v_0}[v_t]\big|\,dt &\lesssim \int_{2^jT2^{-k}}^{\infty} e^{-t/\tau}\,dt \\ &\lesssim e^{-2^{j-1}/\tau}.\end{aligned}$$

We can insert this in the upper bound for the integral to obtain

$$\left\|\int_0^T v_t\,dt\right\|_{L^p} \leq \left(1 + \sum_{j\geq 0} 2^{-j/2} e^{-2^{j-1}/\tau}\right)\sqrt{T}. \tag{8}$$

□

2.4.2. Convergence in Hölder spaces. We are ready to prove Proposition 2.7 on the weak convergence of X^σ in any Hölder space $C^\alpha([0,1],H)$ to the Brownian motion in H with covariance given by formula (7).

Proof of Proposition 2.7. From the tightness result in $C^\alpha([0,1],H)$ stated in Proposition 2.8, it is sufficient to show that X^σ converges weakly in $C^0([0,1],H)$ to the above mentioned Brownian motion. Since we start the velocity process from its invariant measure, the position process X^σ has stationary increments, and any weak limit will have the same property. From Corollary 2.6, the increments of a weak limit are independent on disjoint intervals; continuity of a limit process gives independence of the increments on adjacent intervals. Any weak limit of the X^σ is thus a Brownian motion, and uniqueness will follow from identifying uniquely its covariance. The latter is identified writing

$$\begin{aligned}\mathbb{E}\big[\ell(X_T^\sigma)^2\big] &= \frac{2}{\sigma^4}\int_0^{\sigma^4 T}\int_s^{\sigma^4 T}\mathbb{E}\big[\ell(v_s)\ell(v_t)\big]\,dt\,ds\\ &= 2\int_0^\infty\left(\frac{1}{\sigma^4}\int_0^\infty \mathbf{1}_{s+u\leq\sigma^4 T}ds\right)\mathbb{E}\big[\ell(v_0)\ell(v_u)\big]\,du.\end{aligned}$$

We see on this expression that is has limit

$$2T\int_0^\infty \mathbb{E}\big[\ell(v_0)\ell(v_u)\big]\,du,$$

using the decorrelation estimate from Proposition 2.4 to justify dominated convergence. □

We aim now at improving the weak invariance principle of Proposition 2.7 into a weak invariance principle for the canonical rough path associated with X^σ. This will be crucial in Section 4 when defining kinetic Brownian motion in a diffeomorphism space as the solution of a differential equation driven by X^σ, and proving the interpolation results of Theorem 4.3 and Theorem 4.4 by a continuity argument. We recall in the next section all we need to know from rough paths theory.

2.5. The flavour of rough paths theory

It is not our purpose here to give a detailled account of rough paths theory. We refer the reader to the lecture notes [LCL07, FH14, Bau14, Bai15b], for introductions to the subject from different point of views. The following will be sufficient for our needs here.

Rough paths theory is a theory of ordinary differential equations

$$dz_t = \sum_{i=1}^{\ell} V_i(z_t)\,dh_t^i, \tag{9}$$

controlled by non-smooth signals $h\in C^\alpha([0,1],\mathbb{R}^\ell)$. The point z_t moves here in $\mathbb{R}^d$, where we are given sufficiently regular vector fields V_i. Young integration theory [You36, Lyo94] allows to make sense of the integral $\int_0^\cdot V(y_s)dh_s$, for paths y,h that are α-Hölder, for $\alpha>\frac12$, as an $\mathbb{R}^d$-valued α-Hölder path depending in a locally Lipscthiz way on y and h. This allows to formulate the differential equation (9) as a fixed point problem for a contracting map from $C^\alpha([0,1],\mathbb{R}^d)$ into itself, and to obtain as a consequence the continuous dependence of the solution path on the driving control h. Lyons-Young theory cannot be used for α-Hölder controls with $\alpha<\frac12$, as even in $\mathbb{R}$, with one dimensional controls, there exists no *continuous* bilinear form on $C^\alpha([0,1],\mathbb{R})\times C^\alpha([0,1],\mathbb{R})$ extending the Riemann integral $\int_0^1 y_t dh_t$, of smooth paths y,h; see Propositon 1.29 of [LCL07]. (This can be understood from a Fourier analysis point of view as a consequence of the fact that the resonant operator from Littlewood-Paley theory is unbounded on $C^\alpha([0,1],\mathbb{R})\times C^{\alpha-1}([0,1],\mathbb{R})$, when $2\alpha-1<0$; see [BCD11].) Lyons' deep insight was to realize that what really fixes the dynamics of a solution path to the controlled differential equation (9) is not only the increments dh_t, or $h_t - h_s$, of the control, but rather the increments of h together with the increments of a number of its iterated integrals. This can be understood from the fact that for a smooth control, one has the Taylor-type expansion

$$\begin{aligned} f(z_t) = f(z_s) + \left(\int_s^t dh^i_u\right)(V_i f)(z_s) + \left(\int_{s\leq u_2\leq u_1\leq t} dh^j_{u_2} dh^k_{u_1}\right)\big(V_j V_k f\big)(z_s) \\ + \int_{s\leq u_3\leq u_2\leq u_1\leq t} \big(V_n V_j V_k f\big)(z_{u_3})\, dh^n_{u_3} dh^j_{u_2} dh^k_{u_1}, \end{aligned}$$

for any real-valued smooth function f on $\mathbb{R}^d$. (We use Einstein' summation convention, with integer indices in $[1,\ell]$.) We consider here the vector fields V_i as first order differential operators, so we have for instance

$$V_j V_k f = (D^2 f)(V_j, V_k) + (Df)\big((DV_k)(V_j)\big).$$

The usual first order Euler scheme

$$z_t \simeq z_s + (h^i_t - h^i_s)V_i(z_s),$$

is refined by the above second order Milstein scheme

$$z_t \simeq z_s + (h^i_t - h^i_s)V_i(z_s) + \left(\int_{s\leq u_2\leq u_1\leq t} dh^j_{u_2} dh^k_{u_1}\right)\big(V_j V_k\big)(z_s),$$

whose one step error is given explicitly by the above triple integral, of order $|t-s|^3$, for a C^1 control h. The iterated integrals

$$\int_{s\leq u_2\leq u_1\leq t} dh^j_{u_2} dh^k_{u_1} = \int_{s\leq u_1\leq t} \big(h^j_{u_1} - h^j_s\big)\, dh^k_{u_1},$$

are however meaningless for a control $h \in C^\alpha([0,1],\mathbb{R}^\ell)$, when $\alpha \leq 1/2$. A p-rough path $\mathbf{X}$ above h, with $2 \leq p < 3$, is exactly the datum of h together with a quantity, indexed by $(s \leq t)$, that plays the role of these iterated integrals. Set $[0,1]_\leq := \big\{(s,t) \in [0,1]^2\,;\, s \leq t\big\}$, and recall that $\big(\mathbb{R}^\ell\big)^{\otimes 2}$ stands for the set of $\ell \times \ell$ matrices.

Definition 2.11. *Fix $2 \leq p < 3$. A p-**rough path** $\mathbf{X}$ over $\mathbb{R}^\ell$, is a map*

$$\begin{aligned} [0,1]_\leq &\to \mathbb{R}^\ell \times \big(\mathbb{R}^\ell\big)^{\otimes 2} \\ (s,t) &\mapsto \big(X_{ts}, \mathbb{X}_{ts}\big), \end{aligned}$$

such that

$$X_{ts} = h_t - h_s,$$

for a $C^\alpha([0,1],\mathbb{R}^\ell)$ path h, and $\mathbb{X}$ satisfies Chen's relations

$$\mathbb{X}_{ts} = \mathbb{X}_{tu} + X_{us} \otimes X_{tu} + \mathbb{X}_{us},$$

for all $0 \leq s \leq u \leq t \leq 1$. The $1/p$-Hölder norm on X, and the $2/p$-Hölder norm on $\mathbb{X}$, define jointly a complete metric on the nonlinear space $\mathsf{RP}(p)$ of p-rough paths.

Chen's relation accounts for the fact that for a C^1 path h, one has indeed

$$\begin{aligned}\int_{s\leq u_1\leq t}\big(h^j_{u_1}-h^j_s\big)\,dh^k_{u_1} &= \int_{u\leq u_1\leq t}\big(h^j_{u_1}-h^j_u\big)\,dh^k_{u_1} + \big(h^j_u-h^j_s\big)\big(h^k_t-h^k_u\big)\\ &\quad+\int_{s\leq u_1\leq u}\big(h^j_{u_1}-h^j_s\big)\,dh^k_{u_1}\end{aligned}$$

for any $0\leq s\leq u\leq t\leq 1$, and any indices $1\leq j,k\leq \ell$. One has also in that case, by integration by parts, the identity

$$\begin{aligned}&\int_{s\leq u_1\leq t}\big(h^j_{u_1}-h^j_s\big)\,dh^k_{u_1} + \int_{s\leq u_1\leq t}\big(h^k_{u_1}-h^k_s\big)\,dh^j_{u_1}\\ &= \frac{1}{2}\big(h^j_t-h^j_s\big)\big(h^k_t-h^k_s\big).\end{aligned}$$

A p-rough path $\mathbf{X}$ such that the symmetric part of $\mathbb{X}_{ts}$ is equal to $\frac{1}{2}\,X_{ts}\otimes X_{ts}$, for all times $0\leq s\leq t\leq 1$, is called **weakly geometric**. The set of weakly geometric p-rough paths is closed in $\mathsf{RP}(p)$. For a C^1 path h defined on the time interval $[0,1]$, setting $X_{ts} := h_t - h_s$ and

$$\mathbb{X}_{ts} := \int_s^t X_{us}\otimes dX_u,$$

for all $0\leq s\leq t\leq 1$, defines a weak geometric p-rough path, for any $2\leq p<3$, called the **canonical rough path associated with** h**.** Let B stand for an ℓ-dimensional Brownian motion. The Stratonovich Brownian rough path $\mathbf{B}=(B,\mathbb{B})$ is defined by

$$\mathbb{B}_{ts} := \int_{s\leq u\leq t}(B_u-B_s)\otimes \circ dB_u.$$

It is almost surely a weak geometric p-rough path, for any $2<p<3$.

Definition 2.12. *Let C^3_b vector fields $(V_i)_{1\leq i\leq \ell}$ on $\mathbb{R}^d$ be given, together with a weak geometric p-rough path $\mathbf{X}$ over $\mathbb{R}^\ell$. A path $(z_t)_{0\leq t\leq 1}$ is said to be a solution to the rough differential equation*

$$dz_t = V(z_t)\,d\mathbf{X}_t \tag{10}$$

if there is an exponent $a>1$, such that one has

$$f(z_t) = f(z_s) + X^i_{ts}(V_if)(z_s) + \mathbb{X}^{jk}_{ts}\big(V_jV_kf\big)(z_s) + O\big(|t-s|^a\big), \tag{11}$$

for any smooth real-valued function f on $\mathbb{R}^d$, and any times $0\leq s\leq t\leq 1$.

The above $O(\cdot)$ term is allowed to depend on f. Importantly, the solution of a rough differential equation driven by the Stratonovich Brownian rough path coincides almost surely with the solution of the corresponding Stratonovich differential equation; see e.g. the lecture notes [FH14, Bai14].

Theorem 2.13 (Lyons' universal limit theorem). *The rough differential equation* (10) *has a unique solution. It is an element of $C^{1/p}([0,1],\mathbb{R}^d)$ that depends continuously on $\mathbf{X}$.*

The map that associates to the driving rough path the solution to a given rough differential equation, seen as an element of $C^{1/p}([0,1],\mathbb{R}^d)$, is called the **Itô-Lyons solution map**. If $(\mathbf{X}^n)$ is a sequence of random weak geometric p-rough path in $\mathbb{R}^\ell$, converging weakly to a limit random weak geometric p-rough path $\mathbf{X}$, the continuity of the Itô-Lyons solution map gives for free the weak convergence in $C^{1/p}([0,1],\mathbb{R}^d)$ of the laws of the solutions to Equation (10) driven by the $\mathbf{X}^n$, to the law of the solution of that equation driven by $\mathbf{X}$.

The theory works perfectly well for dynamics with values in Banach spaces or Banach manifolds, and driving rough paths $\mathbf{X} = (X, \mathbb{X})$, with X taking values in a Banach space E. One needs to take care in that setting to the tensor norm used to define the completion of the algebraic tensor space $E \otimes_a E$, as this may produce non-equivalent norms, while this norm is used to define the norm of a rough path. Note that families of vector fields $(V_1, \dots, V_\ell)$ are then replaced in that setting by one forms on E with values in the space of vector fields on the space where the dynamics takes place. See e.g. Lyons' original work [Lyo98] or Cass and Weidner's work [CW16] for the details. See e.g. [Bai15a] for a simple proof of Lyons' universal limit theorem in that general setting.

The vector fields in Definition 2.12 and Theorem 2.13 are required to be C^3_b. This is used to get solution of equation (10) that are defined on the whole time interval $[0,1]$. Only local in time existence results can be obtained when working with generic unbounded vector fields, or on a manifold; see [BC20, Dri18] for global in time well-posedness results. The Taylor-like expansion property (11) defining a solution path is then only required to hold for each time s, for t sufficiently close to s. One still has continuity of the solution path with respect to the driving rough path, in an adapted sense. See e.g. Section 2.4.2 of [ABT15]. This continuity property is sufficient to obtain the local weak convergence of the laws of the solution path to the corresponding limit path, for random driving weak geometric p-rough paths converging weakly to a limit random weak geometric p-rough path. See Definition 4.2 below for the definition of local weak convergence.

So far, we have defined kinetic Brownian motion (x^σ_t, v^σ_t) in H from its unit velocity process v^σ. We have seen in Proposition 2.7 that its time rescaled position process $(X^\sigma_t) := (x^\sigma_{\sigma^2 t})$ converges weakly in $C^\alpha\big([0,1], H\big)$ to a Brownian motion with explicit covariance (7), for any $\alpha < 1/2$. We prove in the next section that the canonical rough path $\mathbf{X}^\sigma$ associated with X^σ converges weakly as a weak geometric p-rough path to the Stratonovich Brownian rough path associated with the Brownian motion with covariance (7), for any $2 < p < 3$. This convergence result will be instrumental in Section 4 to prove that the Cartan development in diffeomorphism spaces of the time rescaled kinetic Brownian motion in Hilbert spaces of vector fields converge to some limit dynamics as σ increases to ∞. This will come as a direct consequence of the continuity of the Itô-Lyons solution map.

Remark 2.14. *The idea of using rough paths theory for proving elementary homogenization results was first tested in the work* [FGL13] *of Friz, Gassiat and Lyons, in their study of the so-called* physical Brownian motion in a magnetic field. *That random process is described as a* C^1 *path* $(x_t)_{0\leq t\leq 1}$ *in* $\mathbb{R}^d$ *modeling the motion of an object of mass* m*, with momentum* $p = m\dot{x}$*, subject to a damping force and a magnetic field. Its momentum satisfies a stochastic differential equation of Ornstein-Uhlenbeck form*

$$dp_t = -\frac{1}{m}\,Mp_t dt + dB_t,$$

for some matrix M*, whose eigenvalues all have positive real parts, and* B *is a d-dimensional Brownian motion. While the process* $(Mx_t)_{0\leq t\leq 1}$ *is easily seen to converge to a Brownian motion* W*, its rough path lift is shown to converge in a rough paths sense in* L^q*, for any* $q \geq 2$*, to a random rough path* different from *the Stratonovich Brownian rough path associated to* W*.*

A number of works have followed this approach to homogenization problems for fast-slow systems; see [ABT15, KM16, KM17, BC17, CFK$^+$19] *for a sample.*

2.6. Rough paths invariance principle for the canonical lift

As in Section 2.4, *we assume in all of this section that the initial condition* v_0 *of the velocity process of kinetic Brownian motion is distributed according to its invariant probability measure* μ*, from Theorem* 2.3.

Let $\mathbf{X}^\sigma = (X^\sigma, \mathbb{X}^\sigma)$ stand for the canonical rough path associated to the random $\mathcal{C}^1$ path X^σ, where we recall that

$$\mathbb{X}^\sigma_{ts} = \int_s^t (X^\sigma_u - X^\sigma_s)\otimes dX^\sigma_u = \frac{1}{\sigma^4}\int_{\sigma^4 s}^{\sigma^4 t}\int_{\sigma^4 s}^{u} v_r\otimes v_u\,drdu.$$

Recall that the tensor space $H\otimes H$ is equipped with its natural complete Hilbert(-Schmidt) norm.

2.6.1. Tightness in rough paths space.

Proposition 2.15. *For any* $p \geq 2$*, we have*

$$\sup_{\sigma>0}\mathbb{E}\big[|\mathbb{X}^\sigma_{t,s}|^p\big] \lesssim |t-s|^p.$$

It follows in particular from Proposition 2.8, Lemma 2.15 and the known Kolmogorov-Lamperti criterion for rough paths that the family of laws $\mathcal{L}(\mathbf{X}^\sigma)$ is tight in $\mathrm{RP}(\alpha^{-1})$, for any $1/3<\alpha<1/2$.

Proof. The statement of the lemma is a consequence of the estimate

$$\mathbb{E}\left[\left|\int_0^T\int_0^t v_s\otimes v_t\,dsdt\right|^p\right] \lesssim_p T^p,$$

for $T\geq 1$; we prove the latter. We use for that purpose the same kind of multiscale martingale/coboundary decomposition as in the proof of Lemma 2.10. Let k the unique integer such that

$$1\leq \delta := T2^{-k} < 2.$$

Define

$$A_j := \int_{j\delta}^{(j+1)\delta} \int_0^t v_s \otimes v_t \, dsdt,$$

and

$$\widehat{\mathcal{F}}_j := \mathcal{F}_{(j+1)\delta} = \sigma\Big(v_s, s \leq (j+1)\delta\Big).$$

As above, we can assume without loss of generality that $\widehat{\mathcal{F}}_j$ is defined for all $j \geq -2^{k+1}$, as v_0 is assumed to be distributed according to the invariant probability measure of the velocity process. Then the integral rewrites as

$$\begin{aligned} &\int_0^T \int_0^t v_s \otimes v_t \, dsdt \\ &\qquad = \Big(A_0 - \mathbb{E}\big[A_0 | \widehat{\mathcal{F}}_{-1}\big]\Big) + \cdots + \Big(A_{2^k-1} - \mathbb{E}\big[A_{2^k-1} | \widehat{\mathcal{F}}_{2^k-2}\big]\Big) \\ &\qquad\quad + \mathbb{E}\big[A_0 | \widehat{\mathcal{F}}_{-1}\big] + \cdots + \mathbb{E}\big[A_{2^k-1} | \widehat{\mathcal{F}}_{2^k-2}\big] \end{aligned} \tag{12}$$

The first sum is a martingale difference with respect to $(\widehat{\mathcal{F}}_n)_{n\geq 0}$, albeit not stationary,

$$\begin{aligned} &\mathbb{E}\Big[\Big| \sum_{0\leq j<2^k} \Big(A_j - \mathbb{E}\big[A_j|\widehat{\mathcal{F}}_{j-1}\big]\Big)\Big|^p\Big]^{\frac{1}{p}} \\ &\qquad \lesssim_p 2^{k/2} \bigg(2^{-k} \sum_{0\leq j<2^k} \mathbb{E}\Big[\big|A_j - \mathbb{E}\big[A_j|\widehat{\mathcal{F}}_{j-1}\big]\big|^p\Big]\bigg)^{\frac{1}{p}} \\ &\qquad \lesssim_p 2^{k/2} \bigg(2^{-k} \sum_{0\leq j<2^k} \mathbb{E}\big[|A_j|^p\big]\bigg)^{\frac{1}{p}}. \end{aligned}$$

Each term is controlled using Lemma 2.8, and the fact that $|v_t| = 1$,

$$\mathbb{E}\big[|A_j|^p\big] \leq \delta^{p-1} \int_{j\delta}^{(j+1)\delta} \mathbb{E}\Big[\Big|\int_0^t v_s \, ds\Big|^p\Big] \, dt \lesssim_p \int_{j\delta}^{(j+1)\delta} t^{p/2} \, dt \lesssim (2^k)^{p/2},$$

so the L^p norm of the first sum in (12) is bounded above by 2^k, up to a constant depending only on p.

The second sum in 12 is treated as in the proof of Lemma 2.10. Set here

$$Z_j^{(n)} := \mathbb{E}\Big[A_{j2^n} + \cdots + A_{j2^n + (2^n-1)} \Big| \widehat{\mathcal{F}}_{j-1}^{(n)}\Big],$$

with

$$\widehat{\mathcal{F}}_j^{(n)} := \widehat{\mathcal{F}}_{(j-1)2^n}.$$

One has

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{0\leq j<2^k}\mathbb{E}\big[A_j\big|\widehat{\mathcal{F}}_{j-1}\big]\Big|^p\Big]^{\frac{1}{p}}\\
&\qquad\lesssim_p \sum_{0\leq n\leq k} 2^{(k-n)/2}\left(\frac{1}{2^{k-n}}\Big(\mathbb{E}\big[\big|Z_0^{(n)}\big|^p\big]+\cdots+\mathbb{E}\big[\big|Z_{2^{k-n}-1}^{(n)}\big|^p\big]\Big)\right)^{\frac{1}{p}},
\end{aligned}$$

and we are left with the study of the moments of the $Z_j^{(n)}$. These variables are the conditional expectation of a double integral, which can be decomposed at time $(j-1)2^n\delta+\delta$ as follows.

$$\begin{aligned}
Z_j^{(n)} &= \mathbb{E}\left[\int_{j2^n\delta}^{(j+1)2^n\delta}\int_0^t v_s\otimes v_t\,dsdt\,\Big|\widehat{\mathcal{F}}_{(j-1)2^n}\right]\\
&= \int_{j2^\ell\delta}^{(j+1)2^n\delta}\int_0^{(j-1)2^n\delta+\delta\vee 0} v_s\otimes\mathbb{E}\Big[v_t\,\Big|\,\widehat{\mathcal{F}}_{(j-1)2^n}\Big]\,dsdt\\
&\quad+\int_{j2^n\delta}^{(j+1)2^n\delta}\mathbb{E}\left[\int_{(j-1)2^n\delta+\delta\vee 0}^t v_s\otimes\mathbb{E}\big[v_t|\mathcal{F}_s\big]\,ds\,\Big|\,\widehat{\mathcal{F}}_{(j-1)2^n}\right]dt\\
&=: R_j^{(n)}+S_j^{(n)}.
\end{aligned}$$

Because the conditioning is from a distant past, the first term is controlled using the exponential mixing and the estimate of Lemma 2.8.

$$\begin{aligned}
\mathbb{E}\Big[\big|R_j^{(n)}\big|^p\Big] &= \mathbb{E}\left[\left|\int_0^{(j-1)2^n\delta+\delta\vee 0} v_s\,ds\right|^p\left|\int_{j2^n\delta}^{(j+1)2^n\delta}\mathbb{E}\big[v_t\,\big|\,\widehat{\mathcal{F}}_{(j-1)2^n}\big]\,dt\right|^p\right]\\
&\lesssim \mathbb{E}\left[\left|\int_0^{(j-1)2^n\delta+\delta\vee 0} v_s\,ds\right|^p\right]\left(\int_{2^n\delta}^{2^{n+1}\delta} e^{-(t-\delta)/\tau}\,dt\right)^p\\
&\lesssim_p \big(2^{k-n}\big)^{\frac{p}{2}}\big(2^n\big)^{\frac{p}{2}}e^{-p2^n/\tau}
\end{aligned}$$

When dealing with the second term, we use the stationarity of v to write

$$\begin{aligned}
\big|S_j^{(n)}\big| &\leq \int_{j2^n\delta}^{(j+1)2^n\delta}\mathbb{E}\left[\int_{(j-1)2^n\delta+\delta}^t\Big|v_s\otimes\mathbb{E}\big[v_t|\mathcal{F}_s\big]\Big|\,ds\,\Big|\,\widehat{\mathcal{F}}_{(j-1)2^n}\right]dt\\
&\overset{\mathcal{L}}{=}\int_{2^n\delta}^{2^{n+1}\delta}\mathbb{E}\left[\int_\delta^t\Big|v_s\otimes\mathbb{E}\big[v_t|\mathcal{F}_s\big]\Big|\,ds\,\Big|\,\widehat{\mathcal{F}}_0\right]dt\\
&\lesssim \int_{2^n\delta}^{2^{n+1}\delta}\mathbb{E}\Big[\int_\delta^t e^{-(t-s)/\tau}\,ds\,\Big|\,\widehat{\mathcal{F}}_0\Big]\,dt\\
&\lesssim 2^n.
\end{aligned}$$

Now we have, for each $0\leq n\leq k$ and $0\leq j<2^{k-n}$,

$$\mathbb{E}\big[\big|Z_j^{(\ell)}\big|^p\big]\lesssim_p \big(2^{k-\ell}\big)^{\frac{p}{2}}\big(2^\ell\big)^{\frac{p}{2}}\cdot e^{-p2^\ell/\tau}+2^{p\ell}$$

so we eventually have

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{0\le j<n}\mathbb{E}\big[A_j\big|\widehat{\mathcal{F}}_{j-1}\big]\right|^p\right]^{\frac{1}{p}} &\lesssim_p \sum_{0\le\ell\le k}\big(2^{k-\ell}2^{\ell/2}e^{-2^\ell/\tau}+2^{(k-\ell)/2}2^\ell\big)\\ &=2^k\sum_{0\le\ell\le k}\big(2^{-\ell/2}\,e^{-2^\ell/\tau}+2^{-(k-\ell)/2}\big)\\ &=2^k\sum_{0\le\ell\le k}2^{-\ell/2}\big(1+e^{-2^\ell/\tau}\big).\end{aligned}$$

This last sum is convergent, so the L^p norm of the second term in (12) is no greater than a constant multiple of 2^k. □

2.6.2. Convergence in rough paths space. We are now ready to state and prove the main result of this section.

Theorem 2.16. *Pick* $1/3<\alpha<1/2$. *The processes* $\mathbf{X}^\sigma$ *converge in law in* $\mathsf{RP}(\alpha^{-1})$*, as* σ *goes to* ∞*, to the Stratonovich Brownian rough path with covariance*

$$C(\ell,\ell')=\int_0^\infty\mathbb{E}\Big[\ell(v_0)\ell'(v_t)+\ell'(v_0)\ell(v_t)\Big]\,dt.$$

Let $\mathbf{X}$ be a random weak geometric α^{-1}-rough path with distribution an arbitrary limit point of the family of laws of the $\mathbf{X}^\sigma$. Write $\mathbf{X}=(B,\mathbb{X})$, with B a Brownian motion with the above covariance. Denote by $\underline{\mathbf{X}}$ the projection of $\mathbf{X}$ on the finite-dimensional space generated by the first d vectors of the basis (e_i) from Section 2.2 – we use below the associated coordinate system. Using a monotone class argument and the tightness result stated in Lemma 2.15, the statement of Theorem 2.16 is a consequence of the following result, given that $d\ge1$ is arbitrary.

Lemma 2.17. *The d-dimensional random rough path* $\underline{\mathbf{X}}$ *is a Stratonovich Brownian rough path with associated covariance matrix* $\mathrm{diag}(\gamma_1,\cdots,\gamma_d)$*, with*

$$\gamma_i:=2\int_0^\infty\mathbb{E}\big[v_0^iv_t^i\big]\,dt.$$

Proof. Let G_d^2 stand for the step-2 nilpotent Lie group over $\mathbb{R}^d$. We prove that the process $(\underline{\mathbf{X}}_{t0})_{0\le t\le1}$ is a G_d^2-valued Brownian motion by showing that it has stationary, independent, increments. The stationarity is inherited from the stationarity of the $\mathbf{X}^\sigma$. The independence of the increments of $\underline{\mathbf{X}}$ on disjoint closed intervals is a consequence of Corollary 2.6 on the convergence to 0 of the mixing coefficient of (v_t). Continuity of $\underline{\mathbf{X}}$ allows to extend the result to adjacent time intervals.

We identify the generator of the G_d^2-valued Brownian motion $(\underline{\mathbf{X}}_t)$ as the generator of the d-dimensional Stratonovich Brownian rough path following the method of [Per18]. We recall the details for the reader's convenience. Note that we only need to consider the joint dynamics of $\underline{B}_t$ and the antisymmetric part $(\underline{\mathbb{A}}_t)$ of $(\underline{\mathbb{X}}_t)$; the former takes values in the Lie algebra $\mathfrak{g}_d^2$ of G_d^2 – a linear space. Denote by $\underline{\mathbb{A}}^B$ the antisymmetric part of Stratonovich Brownian rough path associated with $\underline{B}$. We then have, for any smooth real-valued function f on $\mathbb{R}^d \times \mathfrak{g}_d^2$ with compact support, the identity

$$\begin{aligned}
&\Big(f\big(\underline{B}_t, \underline{\mathbb{A}}_t\big) - f(0)\Big) - \Big(f\big(\underline{B}_t, \underline{\mathbb{A}}_t^B\big) - f(0)\Big) \\
&= (\partial_2 f)(\underline{B}_t, 0)\big(\underline{\mathbb{A}}_t - \underline{\mathbb{A}}_t^B\big) + O\Big(\big|\underline{\mathbb{A}}_t - \underline{\mathbb{A}}_t^B\big|^2\Big) \\
&= \Big((\partial_2 f)(\underline{B}_t, 0) - (\partial_2 f)(0,0)\Big)\big(\underline{\mathbb{A}}_t - \underline{\mathbb{A}}_t^B\big) + (\partial_2 f)(0,0)\big(\underline{\mathbb{A}}_t - \underline{\mathbb{A}}_t^B\big) \\
&\quad + O\Big(\big|\underline{\mathbb{A}}_t - \underline{\mathbb{A}}_t^B\big|^2\Big).
\end{aligned}$$

The conclusion follows by multiplying by t^{-1} and taking expectation, sending t to 0, after recalling that $\underline{\mathbb{A}}_t$ and $\underline{\mathbb{A}}_t^B$ are centered, and recalling the uniform estimates from Proposition 2.15 under the form

$$\big\|\underline{\mathbb{A}}_t\big\|_{L^2} \vee \big\|\underline{\mathbb{A}}_t^B\big\|_{L^2} \lesssim t.$$

□

3. On the geometry of the configuration space

We now complete the second part of the program sketched in Section 1.2, namely we concentrate on the geometric side of the problem. As mentioned above, there is no pre-existing notion of stochastic development on Hilbert manifolds. The construction of such a transport is precisely the object of the present section. More precisely, in the next Sections 3.1 and 3.2, we collect some material on the geometry of the configuration space. Then, i in Sections 3.4 and 3.3, we define parallel transport and Cartan development and show that these notions can be realized as solutions of differential equations in appropriate bundles.

3.1. Configuration space

Let (M, g) be a d-dimensional connected and oriented Riemannian manifold, and $\pi : F \to M$ be a finite-dimensional fiber bundle over M, with vertical bundle $VF \to M$. Think of the trivial bundles $M \times M \to M$, or $M \times TM \to M$, as typical examples. We collect from Palais' seminal work [Pal68] elementary results on the Hilbert manifold $H^s(F)$ of sections of π with Sobolev regularity exponent $s > \frac{d}{2}$.

1. **Sobolev embeddings** hold true, with in particular $H^s(F) \subset C^k(M, F)$, if $s > k + \frac{d}{2}$ and $k \geq 0$.
2. **Variations of H^s-sections of F.** The spaces $TH^s(F)$ and $H^s(VF)$ are isomorphic as Hilbert manifolds. This isomorphism accounts for the fact that an infinitesimal perturbation (δf) of a section f of F, reads as a collection of vertical tangent vectors $(\delta f)(x) \in V_{f(x)}F$, indexed by $x \in M$. As a particular example, for any finite-dimensional manifold N, the spaces $TH^s(M, N)$ and $H^s(M, TN)$ are isomorphic.

3. For any two finite-dimensional fiber bundles F, G above M, the map
$$(f,g) \mapsto \big(x \in M \mapsto (f(x), g(x))\big)$$
is an isomorphism between $H^s(F) \times H^s(G)$ and $H^s(F \times_M G)$.
4. **Omega lemma.** Given a smooth fiber bundle morphism $\Phi : F \to G$, above M, set
$$\omega_\Phi(f) := \Phi \circ f,$$
for any section f of F. Then ω_Φ sends $H^s(F)$ in $H^s(G)$, and $d\omega_\Phi : TH^s(F) \to TH^s(G)$ is isomorphic to $\omega_{d\Phi} : H^s(VF) \to H^s(VG)$, via the isomorphisms $TH^s(F) \simeq H^s(VF)$ and $TH^s(F') \simeq H^s(VG)$.

For $s > \frac{d}{2}$, set
$$\mathscr{M} := H^s(M, M);$$
this will be the **configuration space** of our dynamics. Choosing $s > \frac{d}{2}$, ensures that $\mathscr{M} \subset C^0(M,M)$, by Sobolev embeddings. The tangent space to this Hilbert manifold is given by
$$T\mathscr{M} \simeq H^s(M, TM),$$
from item 2 above. If $s > \frac{d}{2}+1$, elements of $\mathscr{M}$ are C^1 maps from M into itself. Recall in that case from Section 4 of [EM69] that the subset $\mathscr{M}_0$ of $\mathscr{M}$ of H^s maps from M into itself that preserve the volume form by pull-back is then a closed submanifold of $\mathscr{M}$, and that elements of $\mathscr{M}_0$ are diffeomorphisms. So $\mathscr{M}_0$ is a group. We shall always assume implicitly the constraints ($s > \frac{d}{2}$ or $s > \frac{d}{2} + 1$) on the regularity exponent s, when talking about $\mathscr{M}$ or $\mathscr{M}_0$. We recall other elementary facts on $H^s(TM)$ at the end of this section.

To implement a version of Cartan's development machinery in the weak Riemannian setting of the next section, we introduce the following finite-dimensional fiber bundles above M, with M seen below as the first component of each fiber bundle. Given $x, y \in M$, denote by $\mathsf{O}(T_xM, T_yM)$ the set of isometries from T_xM to T_yM. Set
$$F^{(e)} := \Big\{(x,y;e)\,;\,(x,y) \in M \times M, e \in \mathsf{O}(T_xM, T_yM)\Big\},$$
$$F^{(w)} := \Big\{(x,y;w)\,;\,(x,y) \in M \times M, w \in T_xM\Big\},$$
$$F^{(v)} := \Big\{(x,y;v)\,;\,(x,y) \in M \times M, v \in T_yM\Big\},$$
$$F^{(e,v)} := \Big\{\big(x,y;e,v\big)\,;\,(x,y) \in M \times M, e \in \mathsf{O}(T_xM, T_yM), v \in T_yM\Big\}.$$
We understand $H^s(F^{(v)})$ as the set of H^s maps from M into TM, so $T\mathscr{M} \simeq H^s\big(F^{(v)}\big)$. We denote by $\big(\varphi(\cdot), v(\cdot)\big)$ a generic element of $H^s(F^{(v)})$. We have similar interpretations of the other H^s spaces over the corresponding bundles, with similar notations. Since the map
$$F^{(e)} \times_{M\times M} F^{(w)} \to F^{(v)}$$
$$\big((x,y;e),(x,y;w)\big) \mapsto \big(x,y;e(w)\big),$$

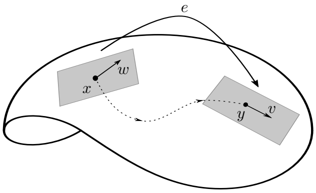

FIGURE 3. An infinitesimal rigid object x is moving along a path. It has position y and velocity v at some time. Its orientation at that time is given by an isometry $e : T_xM \to T_yM$, and its velocity v is given in its initial reference frame by w.

is a smooth bundle morphism, it follows from items 3 and 4 above, that it induces a *smooth map* from $H^s(F^{(w,e)})$ into $H^s(F^{(v)})$. Similarly, the smooth map

$$\begin{aligned} F^{(e,v)} &\to F^{(w)} \\ \big(x, y; e, v\big) &\mapsto \big(x, y; e^{-1}(v)\big), \end{aligned}$$

induces a *smooth map* from $H^s(F^{(e,v)})$ into $H^s(F^{(w)})$.

We refer the reader to the classic textbook [Ros97] for the following elementary facts from functional analysis about the Hodge-Laplace operator Δ on vector fields on M. Note that the Hodge Laplacian is usually defined on forms; identifying vector fields X with the one-forms $X^\flat := \langle X, \cdot\rangle$, we can see it as acting on vector fields. We take the convention that Δ is a non-positive symmetric operator on $L^2(TM)$. This operator has compact resolvant, so one has an eigenspace decomposition

$$L^2(TM) = \bigoplus_{n\geq 0} E_{\lambda_n}, \tag{13}$$

with finite dimensional eigenspaces E_{λ_n}, with corresponding non-positive eigenvalues $\lambda_n \downarrow -\infty$. Eigenvectors of Δ are smooth, from elliptic regularity results. We recover the space $H^s(TM)$ described above setting

$$H^s(TM) = \left\{ f = \sum_{n\geq 0} f_n \in L^2(T\mathcal{M})\,;\, \sum_{n\geq 0} |\lambda_n|^s \|f_n\|_{L^2}^2 < \infty \right\}.$$

The 0-eigenspace is finite dimensional. Any choice of Euclidean norm $\|\cdot\|$ on it defines the topology of $H^s(TM)$, associated with the norm

$$\|f\|_s := \|f_0\| + \left(\sum_{n>0} |\lambda_n|^s \|f_n\|_{L^2}^2 \right)^{1/2}.$$

These result still hold for other Laplacian operators $\widetilde{\Delta} = \Delta + A$, for A a symmetric partial differential operator of order zero. Indeed, one only needs self-adjointness, compactness of the resolvent and elliptic regularity. In particular, according to the Weitzenböck formula, the same results hold for the connection (i.e. Bochner) Laplacian. However the Hodge Laplacian has the advantage for our purposes that it has nice compatibility properties with the divergence. Indeed, if one writes d and δ for the De Rham differential and codifferential, then $(\Delta X)^\flat = -(\delta d + d\delta)(X^\flat)$ and $\operatorname{div} X = -\delta X^\flat$. This will lead to simplifications in the examples we present later.

3.2. Weak Riemannian structure on the configuration space

Denote by $\textsc{Vol}$ the Riemannian volume measure on (M, g), and by $\exp : TM \to M$, its exponential map. The configuration space $\mathscr{M}$ is endowed with a smooth weak Riemannian structure, setting for any $\varphi \in \mathscr{M}$ and $X(\varphi), Y(\varphi) \in T_\varphi \mathscr{M}$,

$$\big(X(\varphi), Y(\varphi)\big)_\varphi := \int_M g_{\varphi(m)}\big(X(\varphi)(m), Y(\varphi)(m)\big)\, \textsc{Vol}(dm). \tag{14}$$

This formula defines by restriction a weak Riemannian metric on the space $\mathscr{M}_0$ of H^s maps from M into itself preserving the volume form. In that setting, notice that if $X(\varphi) = \mathbf{X} \circ \varphi$ and $Y(\varphi) = \mathbf{Y} \circ \varphi$, for some vector fields $\mathbf{X}, \mathbf{Y}$ on M, then the change of variable formula gives

$$\big(X(\varphi), Y(\varphi)\big)_\varphi = \int_M g_m\big(\mathbf{X}(m), \mathbf{Y}(m)\big)\, \textsc{Vol}(dm),$$

so the scalar product is in that case the L^2 scalar product of the vector fields $\mathbf{X}$ and $\mathbf{Y}$. The fact that the topology on $\mathscr{M}$ induced by the scalar product is weaker than the H^s-topology makes non-obvious the existence of a smooth Levi-Civita connection. Ebin and Marsden have proved that

- the L^2 metric (14) is a smooth function on $\mathscr{M}$,
- it has a smooth Levi-Civita connection $\overline{\nabla}$, with associated exponential map Exp well-defined and smooth in a neighbourhood of the zero section; it is explicitly given by

$$\operatorname{Exp}_\varphi(X)(m) = \exp_{\varphi(m)}\big(X(m)\big).$$

The geodesics of $(\mathscr{M}, \overline{\nabla})$ are defined for all times. Denote by ∇ the Levi-Civita connection of (M, g). For smooth right invariant vector fields X, Y on $\mathscr{M}$, with $X(\varphi) = \mathbf{X} \circ \varphi$ and $Y(\varphi) = \mathbf{Y} \circ \varphi$, one has

$$(\overline{\nabla}_X Y)(\varphi) = (\nabla_{\mathbf{X}} \mathbf{Y}) \circ \varphi.$$

The L^2 scalar product is right invariant on the group $\mathscr{M}_0$, from the change of variable formula. The Levi-Civita connection of the L^2 metric on the volume-preserving configuration space $\mathscr{M}_0$ is explicitly given in terms of the Hodge projection operator P on divergence-free vector fields on M. Denote by R_φ the right composition by φ. For any $\varphi \in \mathscr{M}_0$, the map

$$P_\varphi := dR_\varphi \circ P \circ dR_\varphi^{-1}, \tag{15}$$

is indeed the orthogonal projection map from $T_\varphi\mathscr{M}$ into $T_\varphi\mathscr{M}_0$, and its depends smoothly on $\varphi \in \mathscr{M}_0$. So the Levi-Civita connection $\overline{\nabla}^0$ on $\mathscr{M}_0$ is given by

$$\overline{\nabla}^0 = P \circ \overline{\nabla};$$

it is a smooth map [EM69]. Its associated exponential map is no longer given by the exponential map on TM, due to the non-local volume-preserving constraint. Geodesics are not defined for all times anymore. Denote by Id the identity map on M. For smooth right invariant vector fields X, Y on $\mathscr{M}$, with $X(\varphi) = \mathbf{X} \circ \varphi$ and $Y(\varphi) = \mathbf{Y} \circ \varphi$, for vector fields $\mathbf{X}, \mathbf{Y}$ on M, one has

$$\big(\overline{\nabla}^0_X Y\big)(\mathrm{Id}) = P\big(\nabla_{\mathbf{X}}\mathbf{Y}\big).$$

V.I. Arnol'd showed formally in his seminal work [Arn66] that the velocity field $u : [0,T] \to H^s(TM)$ of a geodesic φ_t in $\mathscr{M}_0$, with $u_t := \dot{\varphi}_t \circ \varphi_t^{-1}$, is a solution to Euler's equation for the hydrodynamics of an incompressible fluid. Ebin and Marsden gave an analytical proof of that fact in their seminal work [EM69]. (Besides that classical reference, we refer the reader to Arnold and Khesin's book [AK98], or Smolentsev's thorough review [Smo07] for reference works on the weak Riemannian geometry of the configuration space.)

The flat two-dimensional torus $\mathbf{T}^2$ offers an interesting concrete example. Its symplectic structure allows to identify a Hilbert basis $(A_k, B_k)_{k\in\mathbb{Z}^2\setminus 0}$ of $T_{\mathrm{Id}}\mathscr{M}_0$ from an eigenbasis for the Laplace operator on real-valued functions on $\mathbf{T}^2$; see e.g. Arnold and Khesin's book [AK98], Section 7 of Chap. 1. Denote by ∂_1, ∂_2 the constant vector fields in the coordinate directions, and $k = (k_1, k_2) \in \mathbb{Z}^2$. One has

$$\begin{aligned} A_k &= |k|^{-1}\Big(k_2\cos(k\cdot\theta)\partial_1 - k_1\cos(k\cdot\theta)\partial_2\Big), \\ B_k &= |k|^{-1}\Big(k_2\sin(k\cdot\theta)\partial_1 - k_1\sin(k\cdot\theta)\partial_2\Big). \end{aligned}$$

One can see in the following simulations the image of axis circles by the time 1 map of the associated flow in $\mathbf{T}^2$, corresponding to different inital conditions for u_0, with $\varphi_0 = \mathrm{Id}$. The simulations were done using an elementary finite-dimensional approximation for the dynamics, using the explicit expressions for the Christoffel symbols first given by Arnold in [Arn66].

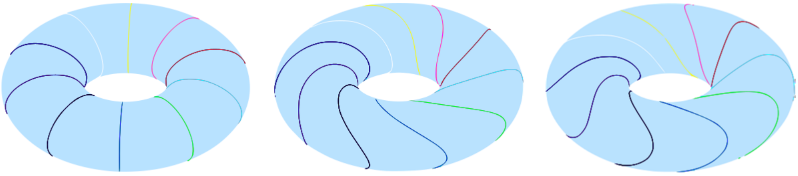

FIGURE 4. Time 1 snapshots of the geodesic flow, for different initial momenta in the volume-preserving diffeomorphism group.

We come back to this point in Section 3.4.

3.3. Parallel transport

We recast in this section the parallel transport operations in $\mathscr{M}$ and $\mathscr{M}_0$, using the bundles F from Section 3.1. This allows to set the notations for the next section on Cartan development operation in $\mathscr{M}$ and $\mathscr{M}_0$. Recall $H^s(F)$ stands for H^s sections from M into the corresponding bundle F. We denote by VF the vertical space in TF, for the canonical projection map $F \to M$. Recall also that $T_{\mathrm{Id}}\mathscr{M}$ is simply the set of H^s vector fields on M.

Denote by $K : TTM \to TM$, the connector associated with the Levi-Civita connection ∇ on M. So, for a path $\gamma_t = (m_t, v_t)$ in TM, one has

$$\nabla_{\dot{m}_t} v_t = K(\dot{\gamma}_t),$$

and

$$\nabla_{\mathbf{X}}\mathbf{Y} = K\big((d\mathbf{Y})(\mathbf{X})\big),$$

for any smooth vector fields $\mathbf{X}, \mathbf{Y}$ on M. The second order tangent bundle $TT\mathscr{M}$ of $\mathscr{M}$ identifies with $H^s(M, TTM)$. The connector $\overline{K}$ associated with the L^2-Levi-Civita connection $\overline{\nabla}$ is given, for a section Y of TTM over an element of $\mathscr{M}$, by

$$\overline{K}(Y) := K \circ Y \in T\mathscr{M}.$$

Set

$$F^{(v,\dot{y})} := F^{(v)} \times_{M\times M} TM.$$

One defines a smooth one form on $F^{(v,\dot{y})}$, with values in $TF^{(v)}$, by requiring that $\nabla_{\dot{y}_t} v_t = 0$ iff

$$\frac{d}{dt}\,(y_t, v_t) = \mathfrak{H}^{(v)}(y_t, v_t; \dot{y}_t).$$

We choose the letter $\mathfrak{H}$, for this horizontal lift of the connection. In simple terms, for any fixed $(y, v) \in TM$, the linear map $\mathfrak{H}^{(v)}(y, v; \cdot)$ identifies the space T_yM to the horizontal subspace of $T_{(y,v)}TM$, via the usual horizontal lift. Note that the definition of $\mathfrak{H}^{(v)}(y, v; \dot{y})$ does not depend on the base point $x \in M$, for a generic element $(x, y; v) \in F^{(v)}$ and $\dot{y} \in T_yM$.

Denote also by $\mathfrak{H}^{(e)}$ the smooth one form on $V_2F^{(v)}$ with values in the space of vector field on $F^{(e)}$, such that for any path $(x, y_t; e_t)$ in $F^{(e)}$, and any vector $w \in T_xM$, the vector $e_t(w) \in T_{y_t}M$ is transported parallely along the M-valued path (y_t) iff

$$\frac{d}{dt}\,(y_t, e_t) = \mathfrak{H}^{(e)}(y_t, e_t; \dot{y}_t).$$

Here again, the base point $x \in M$ is not involved in the definition of the tangent vector $\mathfrak{H}^{(e)}(y, e; \dot{y})$, for a generic element $(x, y; e) \in F^{(e)}$ and $\dot{y} \in T_yM$. Pick

$$(x_0, y_0; e_0) \in F^{(e)},$$

and note that for any vertical vector

$$(\dot{y}, \dot{e}) \in V_{(x_0, y_0; e_0)}F^{(e)},$$

and $v_0 \in T_{y_0}M$, one has

$$(\dot{y}, \dot{e}) = \mathfrak{H}^{(e)}\big(y_0, e_0; v_0\big)$$

iff

$$\big(\dot y, \dot e(w)\big) = \mathfrak{H}^{(v)}\big(y_0, e_0(w); v_0\big) \in V_{(x_0,y_0;e_0(w))}F^{(v)},$$

for any $w \in T_yM$, with $\dot e(w)$ defined naturally. It follows from the Omega Lemma that one defines a *smooth* operator from $H^s\big(F^{(v,\dot y)}\big)$ to $TH^s(F^{(v)})$, setting

$$\overline{\mathfrak{H}}^{(v)}\big(\varphi(\cdot), v(\cdot); \dot\varphi(\cdot)\big) := \mathfrak{H}^{(v)} \circ \big(\varphi(\cdot), v(\cdot); \dot\varphi(\cdot)\big).$$

Similarly, we define a *smooth one-form* on $T_{\mathrm{Id}}\mathscr{M}$ with values in vector fields on $H^s(F^{(e)})$, setting

$$\overline{\mathfrak{H}}^{e}\big(\varphi(\cdot), e(\cdot); \mathbf{X}\big) := \mathfrak{H}^{(e)} \circ \big(\varphi(\cdot), e(\cdot); e(\mathbf{X})\big), \qquad \mathbf{X} \in T_{\mathrm{Id}}\mathscr{M}.$$

Proposition 3.1. *Given a path $\big(\varphi_t(\cdot); e_t(\cdot), v_t(\cdot)\big)_{0\le t\le 1}$ in $H^s(F^{(e,v)})$, one has pointwise*

$$\frac{d}{dt}\big(\varphi_t(x), e_t(x)\big) = \mathfrak{H}^{(e)}\big(\varphi_t(x), e_t(x); v_t(x)\big),$$

for all $x \in M$, iff

$$\frac{d}{dt}\big(\varphi_t, e_t(\mathbf{X})\big) = \overline{\mathfrak{H}}^{(v)}\big(\varphi_t, e_t(\mathbf{X}); v_t\big),$$

for every $\mathbf{X} \in T_{\mathrm{Id}}\mathscr{M}$.

The next two propositions give a description of parallel transport in $\mathscr{M}$ and $\mathscr{M}_0$, respectively, in terms of the vector field $\overline{\mathfrak{H}}^{(v)}$ on $H^s(F^{(v)})$.

Proposition 3.2. *Let $\big(\varphi_t(\cdot), v_t(\cdot)\big)_{0\le t\le 1}$ be a $T\mathscr{M}$-valued path. Then*

$$\overline{\nabla}_{\dot\varphi_t} v_t = 0,$$

iff

$$\frac{d}{dt}\big(\varphi_t, v_t\big) = \overline{\mathfrak{H}}^{(v)}\big(\varphi_t, v_t; \dot\varphi_t\big).$$

Proof. Given $(y, v) \in TM$, the following map identifies T_yM with the vertical subspace of $T_{(y,v)}TM$

$$\mathfrak{V}^{(v)}(y, v; \cdot) : w \in T_yM \mapsto \frac{d}{dt}\Big|_{t=0}(v + tw) \in T_{(y,v)}TM.$$

For any $(x, y; v) \in F^{(v)}$ and $u \in T_{(y,v)}\big(F^{(v)}_x\big)$, one then has

$$u = \mathfrak{H}^{(v)}(y, v; a) + \mathfrak{V}^{(v)}(y, v; b) \quad \text{iff} \quad a = dp_2(u), \text{ and } b = K(u).$$

For an $H^s(F_v)$-valued path $\big(\varphi_t(\cdot), v_t(\cdot)\big)$, one then has the splitting

$$\begin{aligned} \frac{d}{dt}\left(\varphi_t, v_t\right) &= \mathfrak{V}^{(v)} \circ \big(\varphi_t, v_t; K \circ \dot v_t\big)\big) + \mathfrak{H}^{(v)} \circ \big(\varphi_t, v_t; \dot\varphi_t\big) \\ &= \mathfrak{V}^{(v)} \circ \big(\varphi_t, v_t; \overline{\nabla}_{\dot\varphi_t} v_t\big)\big) + \overline{\mathfrak{H}}^{(v)}\big(\varphi_t, v_t; \dot\varphi_t\big). \end{aligned} \tag{16}$$

The result follows because composition by $\mathfrak{V}_v(y, v; \cdot)$ is one-to-one. □

Recall that P stands for Hodge projector on divergence-free vector fields.

Proposition 3.3. *Let* $\big(\varphi_t(\cdot), v_t(\cdot)\big)_{0\leq t\leq 1}$ *be a* $T\mathscr{M}_0$*-valued path. Then*

$$\overline{\nabla}^0_{\dot\varphi_t} v_t = 0,$$

iff

$$\frac{d}{dt}(\varphi_t, v_t) = (dP)\Big(\overline{\mathfrak{H}}^{(v)}(\varphi_t, v_t, ; \dot\varphi_t)\Big).$$

Proof. Write $T_{\mathscr{M}_0}\mathscr{M}$ for the section of $T\mathscr{M}$ above $\mathscr{M}_0$, and write $Q := \mathrm{id} - P : T_{\mathscr{M}_0}\mathscr{M} \to T_{\mathscr{M}_0}\mathscr{M}$, for the projection on the orthogonal in $T\mathscr{M}$ of $T\mathscr{M}_0$. Note that the differential dP of P identifies to P in the fibers, since it is linear. The identification is up to an isomorphism which is exactly the composition by $\mathfrak{V}_v$, in the sense that

$$dP\big(\mathfrak{V}^{(v)}(\varphi, v; v')\big) = \mathfrak{V}^{(v)} \circ \big(\varphi, v; P(v')\big)$$

for any $v, v' \in T_\varphi\mathscr{M}$. As we work with a $T\mathscr{M}_0$-valued path (φ_t, v_t), one has $Q(v_t) = 0$, at all times, so differentiating this identity with respect to t gives

$$dQ(\dot v_t) = 0.$$

Since $P + Q = \mathrm{id}$, we can conclude with the decomposition (16), by rewriting the expression for the time derivative under the form

$$\begin{aligned}\frac{dv_t}{dt} &= dP(\dot v_t) + dQ(\dot v_t)\\ &= \mathfrak{V}^{(v)} \circ \Big(\varphi_t, v_t; P\big(K(\dot v_t)\big)\Big) + dP\big(\mathfrak{H}^{(v)} \circ (\varphi_t, v_t; \dot\varphi_t)\big)\\ &= \mathfrak{V}^{(v)} \circ \Big(\varphi_t, v_t; \overline{\nabla}^0_{\dot\varphi_t} v_t\Big) + dP\Big(\overline{\mathfrak{H}}^{(v)}(\varphi_t, v_t; \dot\varphi_t)\Big).\end{aligned}$$ □

3.4. Cartan and Lie developments

Cartan's moving frame method [Car01] provides a mechanics for constructing C^1 paths on M from C^1 path on $\mathbb{R}^d$, giving something of a chart on pathspace in M. Its description requires the introduction of the orthonormal frame bundle OM over M. It is made up of pairs $z = (m, e)$, with $m \in M$ and e an isometry from $\mathbb{R}^d$ to T_mM. It has a natural finite-dimensional manifold structure, and the Riemannian connection on TM induces vector fields $H_1, \dots, H_d$ on OM by parallel transport of a frame in the direction of its i^{th} direction along the corresponding path in M.

The development in M of a path $(x_t)_{0\leq t\leq 1}$ in $\mathbb{R}^d$ is the natural projection (m_t) in M of the OM-valued path (z_t) solution to the equation

$$\dot z_t = H(z_t)(\dot x_t).$$

Explosion may happen before time 1. This path in M depends not only on m_0 but also on e_0. Conversely, given any C^1 path $(m_t)_{0\leq t\leq 1}$ in M and $z_0 = (m_0, e_0) \in OM$ above m_0, parallel transport of e_0 along the path $(m_t)_{0\leq t\leq 1}$ defines a path $(z_t)_{0\leq t\leq 1}$ in OM, and setting $x_t := \int_0^t e_s^{-1}(\dot m_s)\,ds$, defines a path in $\mathbb{R}^d$ whose Cartan development is $(m_t)_{0\leq t\leq 1}$. Geodesics are Cartan's development of straight lines in $\mathbb{R}^d$.

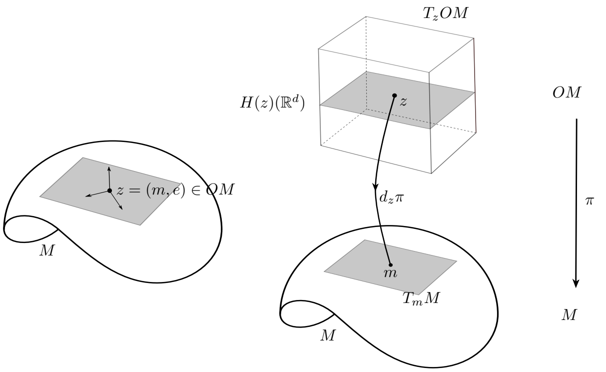

FIGURE 5. For $z \in OM$ and $a = (a_1, \dots, a_d) \in \mathbb{R}^d$, we have $H(z)(a) := \sum_{i=1}^d a_i H_i(z) \in T_zOM$.

We recast the definition of Cartan development given above in a finite-dimensional setting in the following form well suited for the present infinite-dimensional setting.

Definition 3.4. *Let a C^1 path $(\mathbf{X}_t)$ in $T_{\mathrm{Id}}\mathscr{M}$ be given. An $\mathscr{M}$-valued path (φ_t) is the* **Cartan development of** $(\mathbf{X}_t)$ *if there exists a family*

$$e_t : T_{\mathrm{Id}}\mathscr{M} \to T_{\varphi_t}\mathscr{M},$$

of bounded linear maps, with $e_0 = \mathrm{id}$, such that

$$\begin{aligned} &\dot{\varphi}_t = e_t(\dot{\mathbf{X}}_t), \\ &\overline{\nabla}_{\dot{\varphi}_t} e_t(\mathbf{Y}) = 0, \qquad \textit{for all } \mathbf{Y} \in T_{\mathrm{Id}}\mathscr{M}, \end{aligned} \tag{17}$$

at all times where φ_t is well-defined.

This definition conveys the same picture as above. The map e_t, named 'frame', is transported parallely along the path (φ_t), while $\dot{\varphi}_t$ is given by the image by e_t of $\dot{\mathbf{X}}_t$. The existence of a unique Cartan development for a path $(\mathbf{X}_t)$ in $T_{\mathrm{Id}}\mathscr{M}$ is elementary in that case. It follows from Proposition 3.2 that equation (17) is equivalent to requiring that the $H^s\big(F^{(e)}\big)$-valued path (φ_t, e_t) satisfies the equation

$$\frac{d}{dt}(\varphi_t, e_t) = \overline{\mathfrak{H}}^e\big(\varphi_t, e_t; \dot{\mathbf{X}}_t\big). \tag{18}$$

Since the one-form $\overline{\mathfrak{H}}^e$ is smooth, this equation has a unique solution until its possibly finite explosion time.

Here is now the form of Cartan development dynamics in $\mathscr{M}_0$. Recall $T_{\mathrm{Id}}\mathscr{M}_0$ is the set of H^s divergence-free vector fields on M.

Definition 3.5. *Let a C^1 path $(\mathbf{X}_t)$ in $T_{\mathrm{Id}}\mathscr{M}_0$ be given. An $\mathscr{M}_0$-valued path (φ_t) is the* **Cartan development of** $(\mathbf{X}_t)$ *if there exists a family*

$$e_t : T_{\mathrm{Id}}\mathscr{M}_0 \to T_{\varphi_t}\mathscr{M}_0,$$

of bounded linear maps, with $e_0 = \mathrm{id}$*, such that*

$$\begin{aligned} \dot{\varphi}_t &= e_t(\dot{\mathbf{X}}_t), \\ \overline{\nabla}^0_{\dot{\varphi}_t} e_t(\mathbf{Y}) &= 0, \qquad \textit{for all}\, \mathbf{Y} \in T_{\mathrm{Id}}\mathscr{M}_0, \end{aligned} \tag{19}$$

at all times where φ_t *is well-defined.*

The proof of existence of a unique solution to Cartan's development system (19) in $\mathscr{M}_0$ is not fundamentally different from the case of $\mathscr{M}$, and uses Proposition 3.3 instead of Proposition 3.2. It is however more technical, and full details are given in Appendix A. The system is recast as a controlled ordinary differential equation in the state space

$$\mathscr{X} := H^s(F^{(e)}) \times \mathsf{L}\big(H^s(TM)\big),$$

with generic element $\big((\varphi, e), f\big)$, and dynamics of the form

$$\begin{aligned} \frac{d}{dt}\,(\varphi_t, e_t) &= \overline{\mathfrak{H}}^e\Big(\varphi_t, e_t; f_t(\dot{\mathbf{X}}_t)\Big), \\ \frac{d}{dt}\, f_t &= \overline{\mathfrak{H}}^f \left(\frac{d}{dt}\,(\varphi_t, e_t), f_t\right), \end{aligned}$$

driven by a *smooth* vector field-valued one form on $T_{\mathrm{Id}}\mathscr{M}_0$. We use Cartan's development map in the configuration manifolds $\mathscr{M}$ and $\mathscr{M}_0$ in the next section. We conclude this section by a brief comparison between Cartan development and the Lie group notion of development, commonly used to define the stochastic Euler equation.

Let G stand for a finite-dimensional Lie group with Lie algebra $\mathrm{L{\small IE}}(G)$. Lie's development operation provides another way of constructing paths

$$(g_t)_{0\leq t\leq 1}$$

with values in G from paths $(x_t)_{0\leq t\leq 1}$ in $\mathbb{R}^d$, by identifying $T_{g_0}G$ and $\mathbb{R}^d$ via a linear map ι_0, and solving the ordinary differential equation

$$\dot{g}_t = \iota_0(\dot{x}_t)\, g_t.$$

In such a group setting, Malliavin and Airault [AM02] gave a correspondance between the Cartan and Lie notions of development, although this was certainly known to practitioners before; see also [CFM07]. Choose an orthonormal basis of the Lie algebra of G, and denote by $c^n_{k,\ell}$ the structure constants, so the Christoffel symbols are given by $\Gamma^n_{k,\ell} = \frac{1}{2}\left(c^n_{k,\ell} - c^k_{\ell,n} + c^\ell_{n,k}\right)$. Write Γ_k for the antisymmetric endomorphism with matrix $\Gamma^{\cdot}_{k,\cdot}$ in the chosen basis, for $1 \leq k \leq d$, and consider Γ as a linear map from $\mathbb{R}^d$ into the set of antisymmetric endomorphism of the Lie algebra. Denote by $O\mathrm{L{\small IE}}(G)$ the orthonormal group of $\mathrm{L{\small IE}}(G)$.

Proposition 3.6. *Let* $(w_t)_{0\leq t\leq 1}$ *be a* C^1 *path in the Lie algebra of* G*. The path* $(g_t)_{0\leq t\leq 1}$ *solution to the* $\big(O\mathrm{L{\small IE}}(G) \times G\big)$*-valued equation*

$$\begin{aligned} dO_t &:= O_t\, \Gamma(\dot{w}_t)\, dt, \quad O_0 = \mathrm{Id}, \\ dg_t &= O_t(\dot{w}_t) g_t, \end{aligned} \tag{20}$$

is the Cartan development of the path (w_t).

(The system (20) is reminiscent of the equation in

$$H^s(F^{(e)}) \times \mathsf{L}\big(H^s(TM)\big)$$

from Appendix A, recasting Cartan's development dynamics in $\mathscr{M}_0$.) The geodesic started from the identity of G, with direction $\omega \in \textsc{Lie}(G)$, is in particular given in the Lie picture as the solution $(g_t)_{0\leq t\leq 1}$ to the equation

$$\dot{g}_t = \exp\big(t\Gamma(\omega)\big)(\omega)\, g_t.$$

Note that $\exp\big(t\Gamma(\omega)\big)(\omega) \in \textsc{Lie}(G)$. Note also that it is the fact that the Christoffel symbols are constants that allows to reduce the second order differential equation for the geodesics on a generic Riemannian manifold into a first order differential equation, in a Riemannian Lie group setting.

Following Euler's picture, it is this group-oriented point of view that has been considered so far in the geometric viewpoint on fluid hydrodynamics, deterministic or stochastic. The naive implementation of Cartan's machinery in terms of Lie development runs into trouble in the infinite-dimensional setting of $\mathscr{M}$ or $\mathscr{M}_0$. This can be seen on the example of the two dimensional torus and the volume-preserving diffeomorphism group as a consequence of the fact that Christoffel symbols define antisymmetric unbounded operators that have no good exponential in the orthonormal group of $T_{\mathrm{Id}}\mathscr{M}_0$. The problem comes from the fact that $\mathscr{M}$ of $\mathscr{M}_0$ have a *fixed* regularity. See Malliavin's works [Mal99, CFM07] for a quantification of the loss of regularity of Brownian motion in the set of homeomorphisms of the circle, as time increases. The Lie development picture of Cartan's development map can however be used for numerical purposes for simulating kinetic Brownian motion in $\mathscr{M}_0$. It corresponds to having $\dot{w}_t$ a Brownian motion on the unit sphere of the H^s space of divergence-free vector fields on M; see Section 4.

4. Kinetic Brownian motion on the diffeomorphism group

We can now combine the results established in Sections 2 and 3 above, to first define kinetic Brownian motion in configuration space, and then show that it satisfies the desired homogenization property, as the noise parameter σ goes to infinity. Following the same notations as in the last section, let us pick $s > \frac{d}{2}$, or $s > \frac{d}{2} + 1$, depending on whether we work on $\mathscr{M}$ or $\mathscr{M}_0$.

4.1. Kinetic Brownian motion in $\mathscr{M}$

As in Section 3, let (M, g) be a d-dimensional connected and oriented Riemannian manifold. Set $H := H^s(TM)$. As discussed in Section 2.2, any sort of kinetic Brownian motion will have to depend on some covariance structure, given by some operator $\overline{C} : H \to H$. To keep our discussion as intrinsic as possible, we choose it to be some fractional power $(-\Delta)^{-a}$ of the Hodge-Laplace operator, but the reader may choose any other trace-class operator.

On the space of harmonic vector fields, where the Laplacian cannot be inverted, we will take the convention that $(-\Delta)^{-a}$ acts as 0, merely to keep the discussion simple. On the torus, this means that the noise is not strong enough in the large scales that it can move the whole fluid together in one direction. There are again many other ways to choose a covariance so that the noise still has an influence in the large scales, for instance by considering $f(-\Delta)$ in the spectral sense for $f : x \mapsto 1/(1 + x^a)$, or any other function decaying fast enough.

Pick some exponent $a > \frac{d}{2}$, and let $\mathcal{H}$ stand for the L^2 orthogonal of $\ker(\Delta)$ in $H^{s+a}(TM)$, with norm

$$\|f\|_{s+a}^2 = \sum_{n\geq 1} |\lambda_n|^{s+a} \|f_n\|_{L^2}^2,$$

inherited from the eigenspace decomposition (13) of $L^2(TM)$. Let ι stand for the continuous inclusion of $\mathcal{H}$ into H, and use freely the canonical identification of H and H^*. Define $\overline{C}$ as the continuous symmetric operator $\iota\iota^* : H \to H$. With the notations of Section 2.2, we get

$$\alpha_n^2 = |\lambda_n|^{-a},$$

so the operator is trace-class, as a consequence of Weyl's law on a closed manifold. Indeed, it is a general result for closed manifolds that

$$|\lambda_n|^{d/2} \sim n \cdot \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}{d \cdot \mathrm{vol}_g(M)},$$

see Remark 10.4.25. (a) in [Nic21]. Accordingly, it is the covariance of an H-valued Brownian motion W.

We assume that the trace condition

$$3\alpha_1^2 < \mathrm{tr}(\overline{C}), \tag{21}$$

holds true. Note that the faster λ_i goes to ∞, the lesser there is noise in W. The extreme case corresponds to only finitely many non-null α_i. On the other extreme, the bigger the multiplicity of α_1^2 is, the more noise there is in W. The trace condition (21) holds for instance automatically as soon as α_1^2 has multiplicity three.

The Brownian motion v_t^σ on the sphere S of H, associated with the compact injection $\mathcal{H} \hookrightarrow H$, is defined as the solution to the stochastic differential equation

$$dv_t^\sigma = \sigma\, P_{v_t^\sigma}(\circ dW_t),$$

where $P_a : H \to H$, is the orthogonal projection on $\langle a\rangle^\perp$, for any $a \neq 0$, and the position process x_t^σ of kinetic Brownian motion $\left(x_t^\sigma, v_t^\sigma\right)$ in H, given as its integral

$$x_t^\sigma = x_0 + \int_0^t v_s^\sigma\, ds.$$

Kinetic Brownian motion on $\mathscr{M}$ is then defined as Cartan development in $\mathscr{M}$ of the time rescaled kinetic Brownian motion $\left(x_{\sigma^2 t}^\sigma\right)$ in H.

Definition 4.1. *Kinetic Brownian motion on* $\mathscr{M}$ *is the projection* φ_t^σ *on the configuration space* $\mathscr{M}$ *of the solution* $\big(\varphi_t^\sigma, e_t^\sigma\big)$ *to the equation in* $H^s(F^{(e)})$

$$\frac{d}{dt}\left(\varphi_t^\sigma, e_t^\sigma\right) = \overline{\mathfrak{H}}^e\Big(\varphi_t^\sigma, e_t^\sigma; \sigma^2 v_{\sigma^2 t}^\sigma\Big), \tag{22}$$

with initial condition $\varphi_0 = \mathrm{Id}$ *and* $e_0 = \mathrm{Id} \in \mathsf{L}\big(H^s(TM)\big)$.

This equation is only locally well-posed. We introduce the following definition to deal with weak convergence questions for possibly exploding solutions of random or stochastic differential equations. Add a cemetary point $\dagger$ to $H^s(F^{(e)})$, and endow the disjoint union $H^s(F^{(e)}) \sqcup \{\dagger\}$ with its natural topology. Denote by Ω_0 the set of continuous paths $z : [0,1] \to H^s(F^{(e)}) \sqcup \{\dagger\}$, that start from a reference point $z_0 := (\mathrm{Id}, e_0)$ above the identity map on M, and that stay at the cemetery point $\dagger$, if it leaves $H^s(F^{(e)})$. Let $\mathcal{F} := \bigvee_{t\in[0,1]} \mathcal{F}_t$ where $(\mathcal{F}_t)_{0\leq t\leq 1}$ stands for the filtration generated by the canonical coordinate process on pathspace. Let B_R stand for the H^s balls with center z_0 and radius R, for any $R > 0$. The first exit time from B_R is denoted by τ_R, and used to define a measurable map

$$T_R : \Omega_0 \to C\big([0,1], \overline{B}_R\big),$$

which associates to any path $(z_t)_{0\leq t\leq 1} \in \Omega_0$ the path which coincides with z on the time interval $\big[0, \tau_R\big]$, and which is constant, equal to z_{τ_R}, on the time interval $\big[\tau_R, 1\big]$. The following definition then provides a convenient setting for dealing with sequences of random process whose limit may explode.

Definition 4.2. *A sequence* $(\mathbb{Q}_n)_{n\geq 0}$ *of probability measures on* $\big(\Omega_0, \mathcal{F}\big)$ *is said to* **converge locally weakly** *to some limit probability* $\mathbb{Q}$ *if the sequence* $\mathbb{Q}_n \circ T_R^{-1}$ *of probability measures on* $C([0,1], \overline{B}_R)$ *converges weakly to* $\mathbb{Q}\circ T_R^{-1}$, *for every* $R > 0$.

We proved in Theorem 2.16 that the canonical rough path lift $\mathbf{X}^\sigma$ of $\big(x_{\sigma^2 t}^\sigma\big)_{0\leq t\leq 1}$, converges weakly in the space of weak geometric p-rough paths in H, to the Stratonovich Brownian rough path $\mathbf{B} = (B, \mathbb{B})$, with covariance operator

$$C_B(\ell, \ell') = \int_0^\infty \mathbb{E}\Big[\ell(v_0)\ell'(v_t) + \ell'(v_0)\ell(v_t)\Big]\,dt, \quad \ell, \ell' \in H^*.$$

Since one can rewrite Equation (22) as a rough differential equation driven by the rough path $\mathbf{X}^\sigma$

$$\frac{d}{dt}\left(\varphi_t^\sigma, e_t^\sigma\right) = \overline{\mathfrak{H}}^e\Big(\varphi_t^\sigma, e_t^\sigma; d\mathbf{X}_t^\sigma\Big),$$

the continuity of the Itô–Lyons solution map gives the following theorem. Let us recall that the solution of a rough differential equation driven by the Stratonovich Brownian rough path coincides almost surely with the solution of the corresponding Stratonovich differential equation.

Theorem 4.3. *As σ goes to infinity, the $\mathscr{M}$-valued part (φ_t^σ) of kinetic Brownian motion is converging locally weakly to the projection on $\mathscr{M}$ of the $H^s(F^{(e)})$-valued Brownian motion (φ_t, e_t) solution to the stochastic differential equation*

$$\frac{d}{dt}(\varphi_t, e_t) = \overline{\mathfrak{H}}^e\Big((\varphi_t, e_t); \circ dB_t\Big).$$

The motion of φ_t itself is not given as the solution of a stochastic differential equation. This happens already in finite dimension, when defining anisotropic Brownian motion on a d-dimensional Riemannian manifold M as Cartan development of an anisotropic Brownian motion in $\mathbb{R}^d$. One needs the moving orthonormal frame attached to the running point on M, to define the position increment in M from the increment of the driving anisotropic Brownian motion in $\mathbb{R}^d$. The motion in M is in particular non-Markovian, while the motion in OM is Markovian. The same phenomenon happens in the present infinite-dimensional setting, and we do not get here classical semimartingale flows in $H^s(M, M)$ [Kun90], or Brownian flows in critical spaces, such as in Malliavin's work on the canonical Brownian motion on the diffeomorphism group of the circle [Mal99, Fan02, AR02].

We remark here that the stochastic homogenization methods that X.-M. Li used in [Li16] to prove the homogenization result for kinetic Brownian motion in a finite-dimensional, complete, Riemannian manifold, require a positive injectivity radius and a uniform control on the gradient of the distance function over the whole manifold. It is unclear that anything like that is available in the present infinite-dimensional setting, or in the setting of volume-preserving diffeomorphisms investigated in the next section, especially given the fact that $\mathscr{M}$ or $\mathscr{M}_0$ have infinite negative curvature in some directions. The robust pathwise approach of rough paths allows to circumvent these potential issues.

4.2. Kinetic Brownian motion in $\mathscr{M}_0$

Let H_0 stand for the closed subspace of H of divergence-free vector fields on the fluid domain M. It is the tangent space at the identity map of the closed submanifold $\mathscr{M}_0$ of $\mathscr{M}$ of diffeomorphisms that leave invariant the Riemannian volume form of M. As for $\mathscr{M}$, the kinetic Brownian motion in $\mathscr{M}_0$ has to depend on some covariance $\overline{C}_0$. In this case also, we choose $\overline{C}_0$ to be the restriction of Δ^{-a} to H_0. Let us discuss first the spectral theory of Δ over H_0. Because we chose to consider the Hodge Laplacian, we can make use of the classical Hodge decomposition. In particular, $\Delta : H \to H$ leaves H_0 invariant (in fact $\Delta_{\mathrm{LB}}\,\mathrm{div} = \mathrm{div}\,\Delta$, for Δ_{LB} the Laplace-Beltrami operator on functions), and admits a nice spectral decomposition there, as in (13). The eigenvalues over H_0 are a subset of those over H, so they have to grow faster.

Fix $a > d/2$. The intersection $\mathcal{H}_0^{s+a}$ of $\mathcal{H}^{s+a}$ with H_0 is continuously embedded into H_0. If ι_0 stands for this injection, the eigenvalues of the continuous symmetric operator $\iota_0\iota_0^* : H_0 \to H_0$ decrease faster than that of $\iota\iota^*$, so it is trace-class, hence the covariance of an H_0-valued Brownian motion W.

The spectrum of $\overline{C}_0 := \iota_0\iota_0^*$ is explicit in the example of the 2-dimensional torus, with maximal eigenvalue 1, with multiplicity 4. The trace condition (3) thus holds true for any $a > \frac{1}{2}$, in that case. Similarly, the spectrum of the Laplacian operator on vector fields on the 2-dimensional sphere is obtained from the spectrum of the Laplacian operator on real-valued functions on the 2-sphere, as a consequence of its canonical symplectic structure [AS89, Yos97]. Eigenvectors are constant multiples of the complex spherical harmonics, so eigenvalues have multiplicity at least two. Here as well, symmetry properties of the 2-dimensional sphere imply that they have actually multiplicity four, so the trace condition (3) holds for free.

Kinetic Brownian motion (x_t^σ, v_t^σ) in H_0 is defined as above from the associated Brownian motion (v_t^σ) on the sphere S_0 of H_0, and its integral. We prove in Theorem A.3 of Appendix A that the Cartan development φ_t^σ in $\mathscr{M}_0$, of the time rescaled kinetic Brownian motion in H_0 is the $\mathscr{M}_0$-part of the solution $(\varphi_t^\sigma, e_t^\sigma, f_t^\sigma)$, to a controlled ordinary differential equation on

$$\mathscr{X} = H^s(F^{(e)}) \times \mathsf{L}\big(H^s(TM)\big)$$

driven by a *smooth* vector field

$$\begin{aligned} \frac{d}{dt}\,(\varphi_t^\sigma, e_t^\sigma) &= \overline{\mathfrak{H}}^e\Big(\varphi_t^\sigma, e_t^\sigma; f_t^\sigma\big(\sigma^2 v_t^\sigma\big)\Big), \\ \frac{d}{dt}\,f_t^\sigma &= \overline{\mathfrak{H}}^f\left(\frac{d}{dt}\,(\varphi_t^\sigma, e_t^\sigma), f_t^\sigma\right). \end{aligned}$$

Here again, one can rewrite that equation as a rough differential equation driven by the canonical rough path $\mathbf{X}^\sigma$ above the time rescalled position process of kinetic Brownian motion in H_0. The continuity of the Itô-Lyons solution map then gives the following theorem.

Theorem 4.4. *As σ goes to infinity, the $\mathscr{M}_0$-valued part (φ_t^σ) of kinetic Brownian motion in $\mathscr{X}$ is converging locally weakly to the projection (φ_t) on $\mathscr{M}_0$ of a $\mathscr{X}$-valued Brownian motion.*

Here again, the dynamics of φ_t^σ is non-Markovian. Note that since kinetic Brownian motion on $\mathscr{M}_0$ is defined by Cartan development, using the L^2 metric (14), the L^2-size of $\dot\varphi_t^\sigma$ is equal to the L^2-norm of v_t^σ. The metric being right invariant on the group $\mathscr{M}_0$, the Eulerian velocity

$$u_t^\sigma := \dot\varphi_t^\sigma \circ (\varphi_t^\sigma)^{-1},$$

also has the same L^2-norm as v_t^σ. The latter is not preserved a priori; neither is the H^s-norm of u_t^σ, as mentioned above after Proposition 3.6.

Denote by Q^0 the quadratic form on $H^s(TM)$, with matrix

$$\mathrm{diag}\big(|\lambda_n|^{-s}\big)_{n\geq 0},$$

in the orthonormal basis of $H^s(TM)$ associated with the eigenvector decomposition (13) for $-\Delta$ on $L^2(TM)$. For each v in the unit sphere S of $H^s(TM)$, one has $Q^0(v) = \|v\|_{L^2}^2$, and

$$\|v\|_{L^2}^2 \leq \lambda_0^{-s}\|v\|_{H^s}.$$

Since the S-valued diffusion (v_t^σ) is ergodic, each component $(v_t^\sigma)_n$ of v_t^σ, in the decomposition (13), is an ergodic process in the interval $\big(-\lambda_n^{-s/2}, \lambda_n^{-s/2}\big)$. The squared L^2-norm of v_t^σ is also an ergodic process in the interval $(0, \lambda_0^{-s})$. It has invariant measure the image of a constant multiple of the measure with density $1/\|u\|$ with respect to the Gaussian measure in H with covariance $\iota_0\iota_0^*$, by the map

$$u \in H \mapsto Q^0\big(u/\|u\|\big),$$

from Proposition 2.3. This is the invariant measure of the squared L^2-norm of the Eulerian velocity process u_t^σ. We emphasize that this invariant measure is independent of the interpolation parameter $\sigma \in (0, \infty)$. We record part of these facts in the following statement.

Corollary 4.5. *Fix $\sigma \in (0, \infty)$. The L^2-norm of the velocity field u^σ of kinetic Brownian motion is an ergodic process taking values in the interval $(0, \lambda_0^{-s})$, with invariant probability measure the image of a constant multiple of the measure with density $1/\|u\|$ with respect to the Gaussian measure in H with covariance $\iota_0\iota_0^*$, by the map*

$$u \in H \mapsto Q^0\big(u/\|u\|\big).$$

It is desirable to study the homogenization problem for other intrinsically randomly perturbated partial differential equations of geometric nature, such as the KdV, (modified) Camassa-Holm equations, or equations with non-local inertia operator, such as the modified Constantin-Lax-Majda equation [Kol17]. The core technical problem, from the geometric/analytic point of view, is the definition of Cartan development map as the solution map of an ordinary differential equation driven by sufficiently regular vector fields on the configuration space. We took advantage, in the present L^2 setting, of the 'pointwise' character of the associated geometric objects to recast things in terms of the F-bundles of Section 3.1. One may have to proceed differently for other weak metrics. We expect the homogenization results proved in Theorem 4.3 and Theorem 4.4 to have analogues in the setting of the strong, complete, Riemannian metrics of [BV20]. Global in time existence results for kinetic Brownian motion and its limit Brownian motion are expected. We leave these questions for a forthcoming work.

We worked here in the Sobolev setting to make things easier and concentrate on the probabilistic problems, and the implementation of the rough path approach in this infinite-dimensional setting. It is a natural question to ask whether one can run the analysis in the Fréchet setting of smooth diffeomorphisms of M, asking for preservation of the regularity of the initial condition and velocity, as in Ebin-Marsden seminal work – Section 12 in [EM69], under proper assumptions on the noise.

Appendix A. Cartan development in $\mathscr{M}_0$

We prove in this Appendix that Cartan's development system (19) on $\mathscr{M}_0$ can be recast as an ordinary differential equation in $H^s\big(F^{(e)}\big) \times \mathsf{L}\big(H^s(TM)\big)$, driven by a smooth vector field. It has, as a consequence, a unique solution, up to a possibly finite explosion time.

Let $\overline{P} : T\mathscr{M} \to T\mathscr{M}$, stand for a smooth vector bundle morphism that coincides with the Hodge projector P from (15) on $T\mathscr{M}_0$. The existence of such a map follows from the following elementary partition of unity result.

Proposition A.1. *Let* $(\mathcal{O}_i)_{i\in I}$ *be an open cover of* $\mathscr{M}$. *Then there exists a smooth partition of unity subordinated to* $(\mathcal{O}_i)_{i\in I}$.

Set

$$\begin{aligned} \overline{\mathfrak{H}}^f : T\mathrm{H}^s\big(F^{(e)}\big) \times \mathsf{L}\big(H^s(TM)\big) &\to T\mathsf{L}\big(H^s(TM)\big) \\ \left(\frac{d}{dt}_{\big|t=0}\big(\varphi_t(\cdot), e_t(\cdot)\big),\, f\right) &\mapsto \frac{d}{dt}_{\big|t=0}\Big(\mathbf{X} \mapsto e_t^{-1}\Big(\overline{P}\big(e_t(f(\mathbf{X}))\big)\Big)\Big). \end{aligned}$$

The letter $\mathbf{X}$ stands for a generic element of $H^s(TM)$, and

$$T\mathsf{L}\big(H^s(TM)\big) = \mathsf{L}\big(H^s(TM)\big).$$

We give the details of the following elementary result.

Lemma A.2. *The map* $\overline{\mathfrak{H}}^f$ *is well-defined and smooth.*

Proof. It is enough to prove that the map

$$\begin{aligned} H^s(F^{(e)}) \times \mathsf{L}\big(H^s(TM)\big) &\to \mathsf{L}\big(H^s(TM)\big) \\ \Big(\big(\varphi(\cdot), e(\cdot)\big), f\Big) &\mapsto \Big(\mathbf{X} \mapsto e^{-1}\Big(\overline{P}\big(e(f(\mathbf{X}))\big)\Big)\Big) \end{aligned}$$

is smooth. Since the map

$$\begin{aligned} H^s(F^{(e)}) \times \mathsf{L}\big(H^s(TM)\big) \times H^s(TM) &\to H^s(TM) \\ \Big(\big(\varphi(\cdot), e(\cdot)\big), f, \mathbf{X}\Big) &\mapsto e^{-1}\Big(\overline{P}\big(e(f(\mathbf{X}))\big)\Big) \end{aligned}$$

is smooth, the problem reduces to the following question. Let a Banach manifold A and a Hilbert space H, be given together with a smooth map $F : A \times H \to H$, that is linear with respect to its second argument. Denote by a and b generic elements of A. Prove that the curryfication $\mathrm{Cur}\, F : a \in A \mapsto F(a, \cdot) \in \mathsf{L}(H)$ is well-defined and smooth.

Write d for the differential operator. We show that $d(\mathrm{Cur}\, F) = \mathrm{Cur}\,(\partial_a F)$. This will be enough, since we can then bootstrap the construction to show that $d^n(\mathrm{Cur}\, F) = \mathrm{Cur}\,(\partial_a^n F)$, is differentiable for any n. Because the result is local, we can assume without loss of generality that A an open set of a Banach space. Fix $a \in M$, and let $\mathcal{U} \times B(0, \varepsilon)$ be a convex neighbourhood of

$(a,0)$ in $A\times H$, such that $\|\partial_a^2F\|_\infty < 1+\|\partial_a^2F(a,0)\|$. Then for all $b\in\mathcal{U}$ and $|w|<1$, one has

$$\Big|F(b,w)-F(a,w)-\partial_aF(a,w)(b-a)\Big| \leq \frac{|b-a|^2}{2}\|\partial_a^2F\|_\infty\,|w|/\epsilon.$$

The conclusion follows from the fact that we have in particular the estimate

$$\Big\|\mathrm{Cur}F(b)-\mathrm{Cur}F(a)-\mathrm{Cur}(\partial_aF)(a;b-a)\Big\| \leq c\,|b-a|^2,$$

for a positive constant c independent of b. □

Choose now a $\mathcal{C}^1$ path $(\mathbf{X}_t)$ with values in $T_{\mathrm{Id}}\mathscr{M}_0$, and zero initial condition. Let $\big((\varphi_t,e_t),f_t\big)$ be the solution in $H^s(F^{(e)})\times\mathsf{L}\big(H^s(TM)\big)$ of the equation

$$\begin{aligned}\frac{d}{dt}\,(\varphi_t,e_t) &= \overline{\mathfrak{H}}^e\Big(\varphi_t,e_t;e_t\big(f_t(\dot{\mathbf{X}}_t)\big)\Big),\\ \frac{d}{dt}\,f_t &= \overline{\mathfrak{H}}^f\left(\frac{d}{dt}\,(\varphi_t,e_t),f_t\right),\end{aligned}\tag{23}$$

with initial condition $e_0=\mathrm{id}_{T\mathcal{M}}$, and $f_0=\mathrm{id}_{H^s(TM)}$. Since the vector field $(\overline{\mathfrak{H}}^e,\overline{\mathfrak{H}}^f)$ is smooth, equation (23) is locally well-posed, possibly up to a finite explosion time ζ.

Theorem A.3. *The path (φ_t) takes values in $\mathscr{M}_0$, and coincides with the Cartan development of $(\mathbf{X}_t)$. We further have $\dot\varphi_t=e_t\big(f_t(\dot{\mathbf{X}}_t)\big)$, so the dynamics* (23) *does not depend on the extension $\overline{P}$ of the Hodge projector P used in the definition of $\overline{\mathfrak{H}}^f$.*

Proof. Let $\mathbf{Y}\in T_{\mathrm{Id}}\mathscr{M}_0$, be a fixed divergence-free vector field on M. We need to show that

$$\overline{\nabla}^0_{\dot\varphi_t}e_t(\mathbf{Y})=0,$$

on the whole time interval $[0,\zeta)$. From Proposition 3.3, this is equivalent to showing that we have

$$\frac{d}{dt}\Big(\varphi_t,e_t\big(f_t(\mathbf{Y})\big)\Big)=dP\Big(\overline{\mathfrak{H}}^{(v)}\Big(\varphi_t,e_t\big(f_t(\mathbf{Y})\big);\dot\varphi_t\Big)\Big).$$

Look at the function

$$(\varphi,e,\mathbf{Z})\mapsto\big(\varphi,e(\mathbf{Z})\big),$$

from $H^s(F^{(e)})\times T_{\mathrm{Id}}\mathscr{M}$ to $H^s(F^{(v)})$, and set

$$\mathfrak{F}:=\partial_{(\varphi,e)}\Big\{(\varphi,e,\mathbf{Z})\mapsto\big(\varphi,e(\mathbf{Z})\big)\Big\}.$$

We have

$$\begin{aligned}&\frac{d}{dt}\Big(\varphi_t,e_t\big(f_t(\mathbf{Y})\big)\Big)\\ &\quad=\mathfrak{F}\left(\frac{d}{dt}\,(\varphi_t,e_t),f_t(\mathbf{Y})\right)-\mathfrak{F}\left(\frac{d}{dt}\,(\varphi_t,e_t),e_t^{-1}\Big(\overline{P}\big(e_t(f_t(\mathbf{Y}))\big)\Big)\right)\\ &\qquad+d\overline{P}\left(\mathfrak{F}\Big(\frac{d}{dt}\,(\varphi_t,e_t),f_t(\mathbf{Y})\Big)\right).\end{aligned}$$

We prove that $e_t(\mathbf{Y})$ is divergence-free. Define for that purpose the subset $I \subset [0,\zeta)$ of times t such that $e_t(\mathbf{Z})$ is divergence-free for all $\mathbf{Z} \in T_{\mathrm{Id}}\mathscr{M}_0$, and φ_t preserves the volume form. It is a non-empty closed subset of $[0,\zeta)$. Fix $t_0 \in I$. It suffices to prove that t_0 is in the interior of I for a well-chosen extension $\widehat{P}$ of P, possibly different from $\overline{P}$. We choose for $\widehat{P}$ any smooth extension of P defined on a neighbourhood of φ_{t_0}, such that $\widehat{P} \circ \widehat{P} = \widehat{P}$. Set $\widehat{Q} := \mathrm{id} - \widehat{P} : T\mathscr{M} \to T\mathscr{M}$, so for a fixed $\mathbf{Z} \in T_{\mathrm{Id}}\mathscr{M}_0$, the quantity

$$Z_t := \widehat{Q}\big(e_t(f_t(\mathbf{Z}))\big)$$

satisfies the equation

$$\begin{aligned}\frac{d}{dt}\, Z_t &= d\widehat{Q}\left(\mathfrak{F}\left(\frac{d}{dt}\,(\varphi_t, e_t), e_t^{-1}\Big(\widehat{Q}(e_t[f_t(\mathbf{Z})])\Big)\right)\right)\\ &= d\widehat{Q}\left(\mathfrak{F}\Big(\frac{d}{dt}\,(\varphi_t, e_t), e_t^{-1}(Z_t)\Big)\right).\end{aligned}$$

This differential equation satisfies the classical Picard-Lindelöf assumptions, so it has a unique solution with given initial condition. Since $Z_0 = 0$ and the constant zero vector field is a solution to the equation, Z_t is identically zero, and $e_t(\mathbf{Z})$ is divergence-free.

This holds true for any $\mathbf{Z}$, in a time interval independent of $\mathbf{Z}$. It follows in particular that $\dot{\varphi}_t = e_t\big(f_t(\dot{\mathbf{X}}_t)\big)$ is locally divergence-free, and φ_t preserves the volume form, in a neighbourhood of the time t_0. The interval I is thus both closed and open, so $I = [0,\zeta)$. The statement of Theorem A.3 follows, since $P\big(e_t(f_t(\mathbf{Y}))\big) = e_t\big(f_t(\mathbf{Y})\big)$, so we get

$$\begin{aligned}\frac{d}{dt}\,\Big(\varphi_t, e_t\big(f_t(\mathbf{Y})\big)\Big) &= d\overline{P}\left(\mathfrak{F}\Big(\frac{d}{dt}\,(\varphi_t, e_t), f_t(\mathbf{Y})\Big)\right)\\ &= d\overline{P}\left(\frac{d}{ds}_{\big|s=t}\Big(\varphi_s, e_s\big(f_t(\mathbf{Y})\big)\Big)\right)\\ &= dP\Big(\overline{\mathfrak{H}}^{(v)}\Big(\varphi_t, e_t\big(f_t(\mathbf{Y})\big); \dot{\varphi}_t\Big)\Big),\end{aligned}$$

using Proposition 3.1 in the last equality. ☐

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