

# THE NONLOCAL MEAN CURVATURE FLOW OF PERIODIC GRAPHS

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ABSTRACT. We establish the well-posedness of the nonlocal mean curvature flow of order  $\alpha \in (0, 1)$  for periodic graphs on  $\mathbb{R}^n$  in all subcritical little Hölder spaces  $h^{1+\beta}(\mathbb{T}^n)$  with  $\beta \in (0, 1)$ . Furthermore, we prove that if the solution is initially sufficiently close to its integral mean in  $h^{1+\beta}(\mathbb{T}^n)$ , then it exists globally in time and converges exponentially fast towards a constant. The proofs rely on the reformulation of the equation as a quasi-linear evolution problem, which is shown to be of parabolic type by a direct localization approach, and on abstract parabolic theories for such problems.

## 1. INTRODUCTION AND MAIN RESULTS

The nonlocal fractional mean curvature  $H_\alpha(\Sigma)$  of order  $\alpha \in (0, 1)$  for a smooth oriented hypersurface  $\Sigma$  in  $\mathbb{R}^{n+1}$  is a quite recent notion introduced in [10] and given, at a point  $p \in \Sigma$ , by the formula

$$H_\alpha(\Sigma)(p) := \frac{2}{\alpha} \int_{\Sigma} \frac{(q-p) \cdot \nu_\Sigma(q)}{|q-p|^{n+1+\alpha}} d\sigma_\Sigma(q),$$

where  $\nu_\Sigma$  and  $d\sigma_\Sigma$  are the unit normal vector field and the surface measure on  $\Sigma$ , respectively, and  $a \cdot b$  denotes the Euclidean scalar product of two vectors  $a$  and  $b$ . In the particular case of  $\Sigma := \{(x, u(x)) : x \in \mathbb{R}^n\}$  being the graph of a function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ , we may express  $H_\alpha(\Sigma)$  at the point  $p = (x, u(x)) \in \Sigma$  according to the formula

$$H_\alpha(\Sigma)(x, u(x)) := -\frac{2}{\alpha} \int_{\mathbb{R}^n} \frac{\delta_{[x,y]}u - y \cdot \nabla u(x-y)}{[|y|^2 + (\delta_{[x,y]}u)^2]^{\frac{n+1+\alpha}{2}}} dy, \quad x \in \mathbb{R}^n, \quad (1.1)$$

using the short hand notation

$$\delta_{[x,y]}u := u(x) - u(x-y), \quad x, y \in \mathbb{R}^n.$$

Besides this integral representation of the nonlocal mean curvature there is also an alternative expression of  $H_\alpha(\Sigma)$  as a principal value integral (see (4.1) below). Actually, the nonlocal mean curvature may be defined more generally for (sufficiently smooth) open sets in  $\mathbb{R}^{n+1}$ , see [10], and the definition above corresponds to the particular case when the open set is a subgraph. In this general situation the nonlocal mean curvature has been a very active research topic in the recent past, see for instance the review article [27] and the references therein and in [29].

In this research, attention is focused on the nonlocal mean curvature flow

$$V(t) = -H_\alpha(\Sigma(t)), \quad t > 0, \quad \Sigma(0) = \Sigma_0, \quad (1.2)$$

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describing the evolution of the family of graphs  $\{\Sigma(t) : t \geq 0\}$  with

$$\Sigma(t) = \{(x, u(t, x)) : x \in \mathbb{R}^n\}, \quad t \geq 0,$$

for  $u(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ , where the initial graph  $\Sigma_0 := \{(x, u_0(x)) : x \in \mathbb{R}^n\}$  with  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  is given. In (1.2),  $V(t)$  is the normal velocity of  $\Sigma(t)$  given by

$$V(t)(x, u(t, x)) := [(1 + |\nabla u|^2)^{-1/2} \partial_t u](t, x), \quad t > 0, x \in \mathbb{R}^n. \quad (1.3)$$

For the corresponding nonlocal mean curvature flow of sets, different concepts were used to study questions related to existence, uniqueness, regularity, occurrence of neck pinching, and the singular limits when  $\alpha \rightarrow 0$  or  $\alpha \rightarrow 1$  in the setting of viscosity solutions, see, e.g., [10–14, 16–18, 22, 28]. Moreover, various properties of smooth solutions (assuming their existence) to the fractional (volume preserving) mean curvature flow were investigated in [19, 36]. Existence and special features of hypersurfaces with constant and almost-constant nonlocal mean curvature have been established in [7–9, 20, 34].

The local well-posedness of the (volume preserving) nonlocal mean curvature flow of bounded regular sets in the setting of classical solutions was addressed only recently in [29] for initial data parametrizing  $C^{1,1}$ -regular hypersurfaces. Additionally, it is shown in [15] for  $C^{1,1}$ -initial data which are close to a sphere in the  $C^1$ -topology that the corresponding solutions to the volume preserving nonlocal mean curvature flow exist globally and converge exponentially fast to a sphere.

Regarding the nonlocal mean curvature flow (1.2) of (possibly unbounded) graphs the local existence and uniqueness of classical solutions determined by initial data with gradients in  $C^\gamma(\mathbb{R}^n)$ , with  $\gamma > \alpha$ , has recently been established in [6] by means of analytic semigroup theory for quasilinear evolution equations. In both references [6, 29] the authors also showed that the solutions become instantaneously smooth.

In this paper we consider the more specific situation of periodic graphs. Within this framework we reformulate (1.2) as a quasilinear parabolic evolution problem in little Hölder spaces and show that it is well-posed and that the corresponding classical solutions define a smooth semiflow in these spaces, see Theorem 1.1 below. In the setting of classical solutions, this seems to be the first well-posedness result which covers all subcritical little Hölder spaces  $h^{1+\beta}(\mathbb{T}^n)$  with  $\beta \in (0, 1)$ , see Remark 1.2. As in [6] we rely on analytic semigroup theory, but we use a self-contained localization argument to derive the parabolicity of (1.2). The semiflow property also enables us to prove that the solution is smooth with respect to both time and space variables for positive times and to investigate stability issues. Indeed, we shall prove in Theorem 1.3 that if the datum  $u_0$  (defining the initial geometry  $\Sigma_0$ ) is sufficiently close to its integral mean in the phase space, then the solution to (1.2) exists globally in time and converges exponentially fast towards a constant (which possibly depends on  $u_0$ .) In fact, we provide the constant and the exponential rate explicitly. A related convergence result for bounded global solutions in  $C^{1+\gamma}(\mathbb{T}^n)$ , with  $\gamma > \alpha$ , has been recently established by different methods in [15].

**1.1. Main Results.** To be more precise, we first note from (1.1) and (1.3) that the nonlocal mean curvature flow (1.2) can be formulated as a quasilinear evolution problem

$$\frac{du}{dt}(t) = \Phi(u(t))[u(t)], \quad t > 0, \quad u(0) = u_0, \quad (1.4)$$

where

$$\Phi(u)[v](x) := \frac{2}{\alpha}(1 + |\nabla u|^2)^{1/2}(x) \int_{\mathbb{R}^n} \frac{\delta_{[x,y]}v - y \cdot \nabla v(x-y)}{[|y|^2 + (\delta_{[x,y]}u)^2]^{\frac{n+1+\alpha}{2}}} dy, \quad x \in \mathbb{R}^n.$$

Denoting by  $h^s(\mathbb{T}^n)$  the little Hölder space on the torus  $\mathbb{T}^n$  of order  $s \geq 0$  (i.e. the closure of  $C^\infty(\mathbb{T}^n)$  in  $C^s(\mathbb{T}^n)$ , see Section 2.1 below for a precise definition) and choosing the Hölder exponents  $\alpha, \beta, \gamma$  such that

$$\alpha, \beta \in (0, 1) \text{ are arbitrary and } \max\{\alpha, \beta\} < \gamma < \min\{1, \alpha + \beta\}, \quad (1.5)$$

we shall prove that the evolution problem (1.4) is of quasilinear parabolic type in the sense that

$$-\Phi \in C^\infty(h^{1+\beta}(\mathbb{T}^n), \mathcal{H}(h^{1+\gamma}(\mathbb{T}^n), h^{\gamma-\alpha}(\mathbb{T}^n))), \quad (1.6)$$

where the set  $\mathcal{H}(h^{1+\gamma}(\mathbb{T}^n), h^{\gamma-\alpha}(\mathbb{T}^n))$  of negative generators of analytic semigroups is defined in Section 2.1. Property (1.6) enables us to use the quasilinear parabolic theory presented in [1] (see also [33]) in the context of (1.4). As a first main result we obtain the following theorem ensuring the local well-posedness of (1.4) and characterizing the blow-up behavior of solutions which are not global. Moreover, it also states that the solution map defines a smooth semiflow on  $h^{1+\beta}(\mathbb{T}^n)$  and that the solutions are smooth for positive times.

**Theorem 1.1.** *Let  $\alpha, \beta, \gamma$  satisfy (1.5). Then, given  $u_0 \in h^{1+\beta}(\mathbb{T}^n)$ , there exists a unique maximal solution  $u := u(\cdot; u_0)$  to (1.4) such that*

$$u \in C([0, T^+), h^{1+\beta}(\mathbb{T}^n)) \cap C((0, T^+), h^{1+\gamma}(\mathbb{T}^n)) \cap C^1((0, T^+), h^{\gamma-\alpha}(\mathbb{T}^n)),$$

with  $T^+ = T^+(u_0) \in (0, \infty]$  denoting the maximal existence time. Moreover, the following properties hold:

(i) *If  $T^+ < \infty$ , then*

$$\lim_{t \rightarrow T^+} \|u(t)\|_{1+\beta} = \infty.$$

(ii) *The map  $[(t, u_0) \mapsto u(t; u_0)]$  defines a semiflow on  $h^{1+\beta}(\mathbb{T}^n)$  which is smooth in the open set*

$$\{(t, u_0) : u_0 \in h^{1+\beta}(\mathbb{T}^n), 0 < t < T^+(u_0)\} \subset \mathbb{R} \times h^{1+\beta}(\mathbb{T}^n)$$

and

$$u \in C^\infty((0, T^+) \times \mathbb{T}^n). \quad (1.7)$$

(iii) *For  $t \in [0, T^+)$  and  $1 \leq j \leq n$ ,*

$$\|u(t)\|_0 \leq \|u_0\|_0 \quad \text{and} \quad \|\partial_{x_j} u(t)\|_0 \leq \|\partial_{x_j} u_0\|_0.$$

*In the particular case  $\beta > \alpha$  we also have  $\|\partial_t u(t)\|_0 \leq \|\partial_t u(0)\|_0$  for  $t \in [0, T^+)$ .*

In Theorem 1.1 we denote by  $\|\cdot\|_0$  the supremum norm and  $\|\cdot\|_{1+\beta}$  is the Hölder norm on  $h^{1+\beta}(\mathbb{T}^n)$ , see Section 2.1 below. We note from (i) and (iii) that the Hölder-seminorm of the gradient of a non-global solution blows up in the limit  $t \rightarrow T^+$ . Moreover, the additional assumption  $\beta > \alpha$  imposed in (iii) for the estimate  $\|\partial_t u(t)\|_0 \leq \|\partial_t u(0)\|_0$  for  $t \in [0, T^+)$  is needed in order to guarantee that  $\partial_t u(0)$  exists, as the integral (1.1) defining  $H_\alpha(\Sigma)$  does in general not converge when  $u \in C^{1+\beta}(\mathbb{T}^n)$  with  $\beta \leq \alpha$ .

*Remark 1.2.* We point out that if  $u$  is a solution to (1.1), then, for  $\lambda > 0$ , also the mapping  $u_\lambda$  given by

$$u_\lambda(t, x) := \lambda^{-1}u(\lambda^{1+\alpha}t, \lambda x),$$

is a solution to (1.1). This scaling invariance property identifies  $h^1(\mathbb{T}^n)$  as a critical space for the evolution problem (1.1). Hence, Theorem 1.1 covers all subcritical spaces  $h^{1+\beta}(\mathbb{T}^n)$  with  $\beta \in (0, 1)$ .

After having established in Section 2 suitable mapping properties of the operator  $\Phi$  appearing in (1.4), we localize  $\Phi(u)$  in Section 3. We then identify the obtained localized operators as Fourier multipliers and use the Mikhlin-Hörmander theorem to establish their generator properties along with suitable resolvent estimates, see Proposition 3.5. This enables us to establish the parabolicity of (1.4), see Theorem 3.1, and implies thus (1.6), so that Theorem 1.1 is a consequence of the results from [1].

In the second part of the paper we investigate stability properties of the flow. We show that the stationary solutions to (1.4) are constant functions and derive the following stability result stating that any solution to (1.4), starting from an initial datum  $u_0$  that is sufficiently close to its integral mean  $\langle u_0 \rangle$  in  $h^{1+\beta}(\mathbb{T}^n)$ , exists for all times and converges at an exponential rate towards a constant as  $t \rightarrow \infty$ .

**Theorem 1.3.** *Let  $\alpha, \beta, \gamma$  satisfy (1.5). Then, there is  $\omega_0 > 0$  such that for all  $\omega \in (0, \omega_0)$  there are constants  $\varepsilon > 0$  and  $M \geq 1$  with the property that, for all  $u_0 \in h^{1+\beta}(\mathbb{T}^n)$  with*

$$\|u_0 - \langle u_0 \rangle\|_{1+\beta} \leq \varepsilon,$$

*the solution  $u = u(\cdot; u_0)$  provided by Theorem 1.1 exists globally (that is  $T^+(u_0) = +\infty$ ), and there is a further constant  $C(u_0)$  with  $|C(u_0)| \leq \|u_0\|_0$  such that*

$$\|u(t) - C(u_0)\|_{1+\beta} \leq M e^{-\omega t} \|u_0 - \langle u_0 \rangle\|_{1+\beta}, \quad t \geq 0. \quad (1.8)$$

A precise formula for  $C(u_0)$  can be found in the proof of Theorem 1.3, see relation (4.13). Moreover, also the exponential rate  $\omega_0$  is explicitly given in the proof of Theorem 1.3 (by (3.26) with  $a = 0$ ).

The proof of Theorem 1.3 is contained in Section 4 and relies on the principle of linearized stability of quasilinear evolution problems in interpolation spaces [33, Theorem 1.3]. We emphasize that the nonlocal mean curvature flow (1.2) is invariant under vertical translations of graphs and therefore, the linearization of  $\Phi$  at any stationary solution has zero as an eigenvalue. To overcome this difficulty, we introduce in proof of Theorem 1.3 a volume preserving unknown that solves a related quasilinear evolution problem, see (4.5), to which the quasilinear principle of linearized stability established in [33, Theorem 1.3] applies. In this way we derive estimates for the volume preserving unknown which are then used to estimate the actual solution  $u = u(\cdot; u_0)$ .

## 2. NOTATION AND MAPPING PROPERTIES OF $\Phi$

In this section we prove fundamental mapping properties of  $\Phi$  in a suitable functional analytic setting which we now introduce.

**2.1. Functional Analytic Setting.** Let  $E_1$  and  $E_0$  be Banach spaces. We denote the space of bounded linear operators from  $E_1$  to  $E_0$  by  $\mathcal{L}(E_1, E_0)$  and set  $\mathcal{L}(E_0) := \mathcal{L}(E_0, E_0)$ . If  $E_1$  is densely embedded in  $E_0$ , we set (following [2])

$$\mathcal{H}(E_1, E_0) = \{A \in \mathcal{L}(E_1, E_0) : -A \text{ generates an analytic semigroup in } \mathcal{L}(E_0)\}$$

and note from [2, Theorem I.1.2.2] that

$$\mathcal{H}(E_1, E_0) = \bigcup_{\substack{\kappa \geq 1 \\ \omega > 0}} \mathcal{H}(E_1, E_0, \kappa, \omega),$$

where  $\mathcal{H}(E_1, E_0, \kappa, \omega)$  consists of the operators  $A$  with the property that  $\omega + A \in \mathcal{L}(E_1, E_0)$  is an isomorphism and

$$\kappa^{-1} \leq \frac{\|(\lambda + A)x\|}{|\lambda| \cdot \|x\|_{E_0} + \|x\|_{E_1}} \leq \kappa, \quad 0 \neq x \in E_1, \operatorname{Re} \lambda \geq \omega.$$

Given  $k \in \mathbb{N}$ , we denote by  $C^k(\mathbb{T}^n)$  the Banach space consisting of functions on  $\mathbb{R}^n$  which are  $2\pi$ -periodic with respect to each variable and which possess continuous derivatives up to order  $k$ . Letting  $\|\cdot\|_0$  be the supremum norm, the norm on  $C^k(\mathbb{T}^n)$  is as usual defined by

$$\|u\|_k := \max_{|\mu| \leq k} \|\partial^\mu u\|_0.$$

Given  $\gamma \in (0, 1)$ , the Hölder space  $C^{k+\gamma}(\mathbb{T}^n)$  consists of those functions  $u \in C^k(\mathbb{T}^n)$  for which the norm

$$\|u\|_{k+\gamma} := [u]_{k,\gamma} + \|u\|_k$$

is finite, where

$$[u]_{k,\gamma} := \max_{|\mu|=k} \sup_{x \neq y} \frac{|\partial^\mu u(x) - \partial^\mu u(y)|}{|x - y|^\gamma}.$$

Finally, the little Hölder space  $h^{k+\gamma}(\mathbb{T}^n)$  is defined as the closure of the set

$$C^\infty(\mathbb{T}^n) := \bigcap_{m \in \mathbb{N}} C^m(\mathbb{T}^n)$$

of smooth functions in  $C^{k+\gamma}(\mathbb{T}^n)$  and  $h^k(\mathbb{T}^n) := C^k(\mathbb{T}^n)$ . Then  $h^s(\mathbb{T}^n)$  is a Banach algebra for  $s \geq 0$  and a closed subspace of  $C^s(\mathbb{T}^n)$ . If  $s \in \mathbb{N}$ , then  $h^s(\mathbb{T}^n) = C^s(\mathbb{T}^n)$ . However, if  $s \notin \mathbb{N}$ , then the inclusion  $h^s(\mathbb{T}^n) \subset C^s(\mathbb{T}^n)$  is strict. Indeed, given  $s \in (0, 1)$ , any periodic function which is smooth in  $(-2\pi, 2\pi)$  and defined as  $|x|^s$  in a small interval containing zero belongs to  $C^s(\mathbb{T}^n) \setminus h^s(\mathbb{T}^n)$ , see e.g. [31, Section 0.2] where an equivalent definition of the little Hölder spaces is provided.

If  $s_2 > s_1 \geq 0$ , then  $h^{s_2}(\mathbb{T}^n)$  is a densely and continuously embedded subspace of  $h^{s_1}(\mathbb{T}^n)$ , since  $C^{s_2}(\mathbb{T}^n)$  embeds continuously in  $C^{s_1}(\mathbb{T}^n)$ . Moreover,  $C^{s_2}(\mathbb{T}^n)$  is a subspace of  $h^{s_1}(\mathbb{T}^n)$  whenever  $0 \leq s_1 < s_2$ .

The little Hölder spaces are stable with respect to continuous interpolation; that is for  $0 \leq s_1 < s_2$  with  $(1 - \theta)s_1 + \theta s_2 \notin \mathbb{N}$  we have

$$(h^{s_1}(\mathbb{T}^n), h^{s_2}(\mathbb{T}^n))_{\theta, \infty}^0 = h^{(1-\theta)s_1 + \theta s_2}(\mathbb{T}^n), \quad (2.1)$$

where  $(\cdot, \cdot)_{\theta, \infty}^0$  is the continuous interpolation functor of exponent  $\theta \in (0, 1)$ , see [21].

**2.2. Mapping Properties of  $\Phi$ .** The goal of this section is to establish Proposition 2.1 below. The mapping property provided by this proposition together with the generator result for  $\Phi(u)$  established later in Theorem 3.1 are the main steps in the proof of the aforementioned claim (1.6).

**Proposition 2.1.** *If  $\alpha, \beta, \gamma$  satisfy (1.5), then*

$$\Phi \in C^\infty(\mathfrak{h}^{1+\beta}(\mathbb{T}^n), \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))). \quad (2.2)$$

The proof of Proposition 2.1 requires some preliminary investigations and is thus postponed to the end of this section.

We first observe that

$$\Phi(u) = (1 + |\nabla u|^2)^{1/2} A(u), \quad (2.3)$$

where

$$A(u)[v](x) := \frac{2}{\alpha} \int_{\mathbb{R}^n} \frac{\delta_{[x,y]} v - y \cdot \nabla v(x-y)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (\delta_{[x,y]} u / |y|)^2]^{\frac{n+1+\alpha}{2}}} dy, \quad x \in \mathbb{R}^n. \quad (2.4)$$

In order to prove Proposition 2.1, it is useful to view the operator  $A$  as an element of a more general class which we now introduce. Given  $p \in \mathbb{N}$  and a smooth function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ , we define the linear operator  $A_p^F(u_1, u_2)[w_1, \dots, w_p, \cdot]$  by setting

$$\begin{aligned} & A_p^F(u_1, u_2)[w_1, \dots, w_p, v](x) \\ & := \int_{\mathbb{R}^n} \frac{\delta_{[x,y]} v - y \cdot \nabla v(x-y)}{|y|^{n+1+\alpha}} F\left(\frac{\delta_{[x,y]} u_1}{|y|}, \frac{\delta_{[x,y]} u_2}{|y|}\right) \left(\prod_{j=1}^p \frac{\delta_{[x,y]} w_j}{|y|}\right) dy \end{aligned}$$

for  $x \in \mathbb{R}^n$ . Note that  $A(u)[v] = A_0^F(u, 0)[v]$  with  $F(\xi_1, \xi_2) := \frac{2}{\alpha}(1 + \xi_1^2)^{-\frac{n+1+\alpha}{2}}$ .

The mapping property established in Lemma 2.2 is the first essential step in the proof of (2.2).

**Lemma 2.2.** *Given  $p \in \mathbb{N}$ , mappings  $u_1, u_2, w_j \in C^{1+\beta}(\mathbb{T}^n)$ ,  $1 \leq j \leq p$ , and a smooth function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ , there exists a positive constant  $C = C(p, F, \|u_1\|_{1+\beta}, \|u_2\|_{1+\beta})$  such that*

$$\|A_p^F(u_1, u_2)[w_1, \dots, w_p, \cdot]\|_{\mathcal{L}(C^{1+\gamma}(\mathbb{T}^n), C^{\gamma-\alpha}(\mathbb{T}^n))} \leq C \prod_{j=1}^p \|w_j\|_{1+\beta}. \quad (2.5)$$

*Proof.* Given  $v \in C^{1+\gamma}(\mathbb{T}^n)$ , we set for  $x, y \in \mathbb{R}^n$  with  $y \neq 0$

$$Q(x, y) := \frac{\delta_{[x,y]} v - y \cdot \nabla v(x-y)}{|y|^{n+1+\alpha}} F\left(\frac{\delta_{[x,y]} u_1}{|y|}, \frac{\delta_{[x,y]} u_2}{|y|}\right) \left(\prod_{j=1}^p \frac{\delta_{[x,y]} w_j}{|y|}\right).$$

Using the mean value theorem, we then have

$$\begin{aligned} |\delta_{[x,y]} v - y \cdot \nabla v(x-y)| &= \left| \int_0^1 y \cdot (\nabla v(x-y+sy) - \nabla v(x-y)) ds \right| \\ &\leq C \min\{\|v\|_1 |y|, [v]_{1,\gamma} |y|^{1+\gamma}\} \\ &\leq C \|v\|_{1+\gamma} \min\{|y|, |y|^{1+\gamma}\}, \end{aligned} \quad (2.6)$$

hence

$$|Q(x, y)| \leq C \|v\|_{1+\gamma} \left( \prod_{j=1}^p \|w_j\|_1 \right) \left( \frac{1}{|y|^{n+\alpha}} \mathbf{1}_{\{|y| \geq 1\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma}} \mathbf{1}_{\{|y| < 1\}}(y) \right),$$

where  $C = C(F, \|u_1\|_1, \|u_2\|_1)$ , and therefore

$$\left| \int_{\mathbb{R}^n} Q(x, y) dy \right| \leq C \|v\|_{1+\gamma} \left( \prod_{j=1}^p \|w_j\|_1 \right) \left( \int_{\{|y| \geq 1\}} \frac{1}{|y|^{n+\alpha}} dy + \int_{\{|y| < 1\}} \frac{1}{|y|^{n+\alpha-\gamma}} dy \right),$$

the latter integrals being both finite since  $0 < \alpha < \gamma$ . Hence,  $A_p^F(u_1, u_2)[w_1, \dots, w_p, v]$  is well-defined,  $2\pi$ -periodic with respect to each variable, and

$$\|A_p^F(u_1, u_2)[w_1, \dots, w_p, v]\|_0 \leq C \|v\|_{1+\gamma} \left( \prod_{j=1}^p \|w_j\|_1 \right). \quad (2.7)$$

We next prove that  $A_p^F(u_1, u_2)[w_1, \dots, w_p, v] \in C^{\gamma-\alpha}(\mathbb{T}^n)$ . Let therefore  $x_1, x_2 \in \mathbb{R}^n$  and  $y \neq 0$  be given. We may estimate

$$|Q(x_1, y) - Q(x_2, y)| \leq C \left[ \left( \prod_{j=1}^p \|w_j\|_1 \right) J_1(y) + J_2(y) \right],$$

where  $C = C(F, \|u_1\|_1, \|u_2\|_1)$  and

$$\begin{aligned} J_1(y) &:= \frac{|\delta_{[x_1, y]} v - y \cdot \nabla v(x_1 - y) - [\delta_{[x_2, y]} v - y \cdot \nabla v(x_2 - y)]|}{|y|^{n+1+\alpha}}, \\ J_2(y) &:= \frac{|\delta_{[x_2, y]} v - y \cdot \nabla v(x_2 - y)|}{|y|^{n+1+\alpha}} \left[ \left( \prod_{j=1}^p \|w_j\|_1 \right) \sum_{i=1}^2 \left| \frac{\delta_{[x_1, y]} u_i}{|y|} - \frac{\delta_{[x_2, y]} u_i}{|y|} \right| \right. \\ &\quad \left. + \sum_{j=1}^p \left( \prod_{\substack{\ell=1 \\ \ell \neq j}}^p \|w_\ell\|_1 \right) \left| \frac{\delta_{[x_1, y]} w_j}{|y|} - \frac{\delta_{[x_2, y]} w_j}{|y|} \right| \right]. \end{aligned}$$

Similarly as in (2.6), we deduce that

$$\begin{aligned} J_1(y) &\leq \frac{1}{|y|^{n+\alpha}} \int_0^1 |\nabla v(x_1 - y + sy) - \nabla v(x_1 - y) - [\nabla v(x_2 - y + sy) - \nabla v(x_2 - y)]| ds \\ &\leq C \|v\|_{1+\gamma} \left( \frac{|x_1 - x_2|^\gamma}{|y|^{n+\alpha}} \mathbf{1}_{\{|y| \geq |x_1 - x_2|\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma}} \mathbf{1}_{\{|y| < |x_1 - x_2|\}}(y) \right), \end{aligned} \quad (2.8)$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^n} J_1(y) dy &\leq C \|v\|_{1+\gamma} \left( \int_{\{|y| \geq |x_1 - x_2|\}} \frac{|x_1 - x_2|^\gamma}{|y|^{n+\alpha}} dy + \int_{\{|y| < |x_1 - x_2|\}} \frac{1}{|y|^{n+\alpha-\gamma}} dy \right) \\ &\leq C \|v\|_{1+\gamma} |x_1 - x_2|^{\gamma-\alpha}. \end{aligned} \quad (2.9)$$

Moreover,

$$\begin{aligned} J_2(y) &\leq \frac{1}{|y|^{n+\alpha}} \left( \int_0^1 |\nabla v(x_2 - y + sy) - \nabla v(x_2 - y)| ds \right) \\ &\quad \times \left[ \left( \prod_{j=1}^p \|w_j\|_1 \right) \sum_{i=1}^2 \int_0^1 |\nabla u_i(x_1 - y + sy) - \nabla u_i(x_2 - y + sy)| ds \right. \\ &\quad \left. + \sum_{j=1}^p \left( \prod_{\substack{\ell=1 \\ \ell \neq j}}^p \|w_\ell\|_1 \right) \int_0^1 |\nabla w_j(x_1 - y + sy) - \nabla w_j(x_2 - y + sy)| ds \right] \end{aligned}$$

and, since  $\beta > \gamma - \alpha$ , we deduce from (2.6) that

$$J_2(y) \leq C \left( \prod_{j=1}^p \|w_j\|_{1+\beta} \right) \|v\|_{1+\gamma} |x_1 - x_2|^{\gamma-\alpha} \left( \frac{1}{|y|^{n+\alpha}} \mathbf{1}_{\{|y| \geq 1\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma}} \mathbf{1}_{\{|y| < 1\}}(y) \right),$$

with  $C = C(p, F, \|u_1\|_{1+\beta}, \|u_2\|_{1+\beta})$ . Thus we have

$$\int_{\mathbb{R}^n} J_2(y) dy \leq C \left( \prod_{j=1}^p \|w_j\|_{1+\beta} \right) \|v\|_{1+\gamma} |x_1 - x_2|^{\gamma-\alpha}. \quad (2.10)$$

Gathering (2.7)-(2.10), we conclude (2.5).  $\square$

Given Banach spaces  $X$  and  $Y$ , we denote by  $C^{1-}(X, Y)$  the space of locally Lipschitz maps from  $X$  to  $Y$ , and  $C^\infty(X, Y)$  is its subspace consisting of the smooth maps from  $X$  to  $Y$ . Moreover,  $\mathcal{L}_{\text{sym}}^p(X, Y)$ , with  $p \in \mathbb{N}$ , stands for the space of  $p$ -linear, bounded, and symmetric maps  $A : X^p \rightarrow Y$ .

In order to prove the mapping property (2.2) we introduce a particular subclass of integral operators of the class  $A_p^F$  introduced above. Namely, given  $p \in \mathbb{N}$  and a smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we set

$$A_p^f(u)[w_1, \dots, w_p, v](x) := \int_{\mathbb{R}^n} \frac{\delta_{[x,y]} v - y \cdot \nabla v(x - y)}{|y|^{n+1+\alpha}} f\left(\frac{\delta_{[x,y]} u}{|y|}\right) \left( \prod_{j=1}^p \frac{\delta_{[x,y]} w_j}{|y|} \right) dy \quad (2.11)$$

for  $x \in \mathbb{R}^n$ . Note that

$$A(u)[v] = A_0^f(u)[v] \quad \text{with} \quad f(s) = \frac{2}{\alpha} (1 + s^2)^{-\frac{n+1+\alpha}{2}}. \quad (2.12)$$

Moreover, we have  $A_p^f(u) = A_p^F(u, 0)$  for  $F(\xi_1, \xi_2) := f(\xi_1)$ . Therefore, we infer from Lemma 2.2 that

$$[u \mapsto A_p^f(u)] : C^{1+\beta}(\mathbb{T}^n) \rightarrow \mathcal{L}_{\text{sym}}^p(C^{1+\beta}(\mathbb{T}^n), \mathcal{L}(C^{1+\gamma}(\mathbb{T}^n), C^{\gamma-\alpha}(\mathbb{T}^n)))$$

is a well-defined mapping. The next lemma shows that this mapping is also Fréchet differentiable and its Fréchet derivative is defined by an integral operator from the same subclass (with  $f'$  denoting the derivative of  $f$ ).

**Lemma 2.3.** *Given  $p \in \mathbb{N}$  and a smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the mapping*

$$[u \mapsto A_p^f(u)] : C^{1+\beta}(\mathbb{T}^n) \rightarrow \mathcal{L}_{\text{sym}}^p(C^{1+\beta}(\mathbb{T}^n), \mathcal{L}(C^{1+\gamma}(\mathbb{T}^n), C^{\gamma-\alpha}(\mathbb{T}^n)))$$

*defined in (2.11) is Fréchet differentiable and*

$$\partial A_p^f(u)[\tilde{u}][w_1, \dots, w_p, v] = A_{p+1}^{f'}(u)[\tilde{u}, w_1, \dots, w_p, v] \quad (2.13)$$

*for all  $u, \tilde{u}, w_j \in C^{1+\beta}(\mathbb{R}^n)$ ,  $1 \leq j \leq p$ , and  $v \in C^{1+\gamma}(\mathbb{R}^n)$ .*

*Proof.* Given  $u, \tilde{u}, w_j \in C^{1+\beta}(\mathbb{R}^n)$ ,  $1 \leq j \leq p$ , with  $\|\tilde{u}\|_{1+\beta} \leq 1$ , and  $v \in C^{1+\gamma}(\mathbb{T}^n)$ , we have

$$\begin{aligned} & (A_p^f(u + \tilde{u})[w_1, \dots, w_p, v] - A_p^f(u)[w_1, \dots, w_p, v] - A_{p+1}^{f'}(u)[\tilde{u}, w_1, \dots, w_p, v])(x) \\ &= \int_{\mathbb{R}^n} R(x, y) \frac{\delta_{[x,y]}v - y \cdot \nabla v(x-y)}{|y|^{n+1+\alpha}} \left( \prod_{j=1}^p \frac{\delta_{[x,y]}w_j}{|y|} \right) dy, \end{aligned}$$

where, given  $x, y \in \mathbb{R}^n$  with  $y \neq 0$ , the fundamental theorem of calculus yields

$$\begin{aligned} R(x, y) &:= f\left(\frac{\delta_{[x,y]}(u + \tilde{u})}{|y|}\right) - f\left(\frac{\delta_{[x,y]}u}{|y|}\right) - \frac{\delta_{[x,y]}\tilde{u}}{|y|} f'\left(\frac{\delta_{[x,y]}u}{|y|}\right) \\ &= \left(\frac{\delta_{[x,y]}\tilde{u}}{|y|}\right)^2 \int_0^1 \int_0^1 s f''\left(\frac{\delta_{[x,y]}(u + \tau s \tilde{u})}{|y|}\right) d\tau ds. \end{aligned}$$

Introducing the smooth function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$F(\xi_1, \xi_2) := \int_0^1 \int_0^1 s f''(\xi_1 + \tau s \xi_2) d\tau ds,$$

we get

$$\begin{aligned} & A_p^f(u + \tilde{u})[w_1, \dots, w_p, v] - A_p^f(u)[w_1, \dots, w_p, v] - A_{p+1}^{f'}(u)[\tilde{u}, w_1, \dots, w_p, v] \\ &= A_{p+2}^F(u, \tilde{u})[\tilde{u}, \tilde{u}, w_1, \dots, w_p, v]. \end{aligned}$$

Keeping  $u$  fixed, we now infer from Lemma 2.2 that there exists a positive constant  $C$  such that for all functions  $\tilde{u} \in C^{1+\beta}(\mathbb{T}^n)$  with  $\|\tilde{u}\|_{1+\beta} \leq 1$  and all  $w_j \in C^{1+\beta}(\mathbb{R}^n)$ ,  $1 \leq j \leq p$ , we have

$$\begin{aligned} & \|A_p^f(u + \tilde{u})[w_1, \dots, w_p, v] - A_p^f(u)[w_1, \dots, w_p, v] - A_{p+1}^{f'}(u)[\tilde{u}, w_1, \dots, w_p, v]\|_{\gamma-\alpha} \\ & \leq C \|v\|_{1+\gamma} \|\tilde{u}\|_{1+\beta}^2 \prod_{j=1}^p \|w_j\|_{1+\beta}, \end{aligned}$$

and the claim follows.  $\square$

We are now in a position to provide the proof of Proposition 2.1.

*Proof of Proposition 2.1.* We first recall that  $\Phi(u) = (1 + |\nabla u|^2)^{1/2} A(u)$ , see (2.3), where, as a direct consequence of Lemma 2.3 and (2.12), we have

$$A \in C^\infty(C^{1+\beta}(\mathbb{T}^n), \mathcal{L}(C^{1+\gamma}(\mathbb{T}^n), C^{\gamma-\alpha}(\mathbb{T}^n))).$$

If  $u, v \in C^\infty(\mathbb{T}^n)$ , the latter property implies that  $A(u)[v] \in C^{\gamma-\alpha+\varepsilon}(\mathbb{T}^n)$  for all  $\varepsilon \in (0, 1-\gamma)$ . But, as  $C^{\gamma-\alpha+\varepsilon}(\mathbb{T}^n)$  is subspace of  $\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n)$ , we have  $A(u)[v] \in \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n)$ . The smoothness of  $A$  now ensures that

$$A \in C^\infty(\mathfrak{h}^{1+\beta}(\mathbb{T}^n), \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))).$$

Finally, observing that

$$[u \mapsto (1 + |\nabla u|^2)^{1/2}] \in C^\infty(\mathfrak{h}^{1+\beta}(\mathbb{T}^n), \mathfrak{h}^\beta(\mathbb{T}^n))$$

and taking into account that  $\beta > \gamma - \alpha$ , we obtain the desired regularity property for  $\Phi$ .  $\square$

### 3. THE PARABOLICITY PROPERTY

Throughout this section we again assume that the Hölder exponents  $\alpha, \beta, \gamma$  satisfy (1.5). The main goal is to show that, given  $u \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$ , the operator  $\Phi(u)$ , viewed as an unbounded operator in  $\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n)$  with domain of definition  $\mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , generates an analytic semigroup in  $\mathcal{L}(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$ . This is the content of the next theorem.

**Theorem 3.1.** *If  $u \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$ , then  $-\Phi(u) \in \mathcal{H}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$ .*

The proof of Theorem 3.1 requires some preparation and is therefore postponed to the end of this section. As a starting point we define the continuous path

$$[\tau \mapsto \Phi(\tau u)] : [0, 1] \rightarrow \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$$

connecting  $\Phi(u)$  to the operator  $\Phi(0)$ . In order to establish Theorem 3.1, we follow a strategy inspired by [23, 24, 26]. In a first step we approximate locally the operator  $\Phi(\tau u)$  by certain operators with “frozen”  $u$ , see Proposition 3.2 below. As a second step we identify these operators as Fourier multipliers and use the Mihlin-Hörmander theorem to establish their generator properties along with (uniform) resolvent estimates, see (3.24) in Proposition 3.5. Together with Proposition 3.2 this builds the core of the proof of Theorem 3.1.

**3.1. Localization of  $\Phi(\tau u)$ .** Guided by (2.4), we start by introducing the class of operators

$$A^a[v](x) := \frac{2}{\alpha} \int_{\mathbb{R}^n} \frac{\delta_{[x,y]} v - y \cdot \nabla v(x-y)}{|y|^{n+1+\alpha}} \frac{1}{[1 + \tau^2(|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \quad (3.1)$$

for  $a \in \mathbb{R}^n$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $x \in \mathbb{R}$ . It is straightforward to infer from the arguments used in the first part of the proof of Lemma 2.2 that  $A^a \in \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$  with

$$\sup_{a \in \mathbb{R}^n} \|A^a\|_{\mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))} < \infty. \quad (3.2)$$

We shall see later on that  $A^a$  is a Fourier multiplier and that its symbol can be expressed explicitly in terms of an integral, see (3.22)-(3.23) below. It is worthwhile to point out the relation

$$A^0 = \Phi(0). \quad (3.3)$$

In order to locally approximate the operator  $\Phi(\tau u)$ ,  $\tau \in [0, 1]$ , by Fourier multipliers  $\delta A^a$  (with  $\delta \geq 1$  and  $a \in \mathbb{R}^n$ ), we choose for each  $\varepsilon \in (0, 1)$  a finite  $\varepsilon$ -localization family; that is,

a family  $\{(\pi_j^\varepsilon, x_j^\varepsilon) : 0 \leq j \leq N\} \subset C^\infty(\mathbb{T}^n, [0, 1]) \times \mathbb{R}$ , with  $N = N(\varepsilon) \in \mathbb{N}$  sufficiently large such that

- $\text{supp } \pi_j^\varepsilon = \overline{\mathbb{B}_\varepsilon(x_j^\varepsilon)} + 2\pi\mathbb{Z}^n$  for  $0 \leq j \leq N$ ,
- $\sum_{j=0}^N \pi_j^\varepsilon = 1$  on  $\mathbb{T}^n$ .

Here and below  $\mathbb{B}_r(x)$  denotes the ball centered at  $x \in \mathbb{R}^n$  with radius  $r > 0$ , and  $\overline{\mathbb{B}_r(x)}$  is its closure. With such a finite  $\varepsilon$ -localization family we associate a family

$$\{\chi_j^\varepsilon : 0 \leq j \leq N\} \subset C^\infty(\mathbb{T}^n, [0, 1])$$

satisfying

- $\chi_j^\varepsilon = 1$  on  $\text{supp } \pi_j^\varepsilon$ ,
- $\text{supp } \chi_j^\varepsilon \subset \overline{\mathbb{B}_{3\varepsilon}(x_j^\varepsilon)} + 2\pi\mathbb{Z}^n$  for  $0 \leq j \leq N$ .

It readily follows from the above properties that, for each  $r \geq 0$ , the map

$$\left[ f \mapsto \sum_{j=0}^N \|\pi_j^\varepsilon f\|_r \right] : \mathfrak{h}^r(\mathbb{T}^n) \rightarrow [0, \infty) \quad (3.4)$$

defines an equivalent norm on  $\mathfrak{h}^r(\mathbb{T}^n)$ . We are now in a position to present the aforementioned localization result.

**Proposition 3.2.** *Let  $\gamma' \in (\max\{\alpha, \beta\}, \gamma)$  and let  $u \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$  and  $\nu > 0$  be given. Then, if  $\varepsilon \in (0, 1)$  is sufficiently small, there exists a positive constant  $K = K(\varepsilon)$  such that for all  $\tau \in [0, 1]$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$  we have*

$$\|\pi_j^\varepsilon \Phi(\tau u)[v] - (1 + \tau^2 |\nabla u|^2)^{1/2} (x_j^\varepsilon) A^{\tau \nabla u(x_j^\varepsilon)} [\pi_j^\varepsilon v]\|_{\gamma-\alpha} \leq \nu \|\pi_j^\varepsilon v\|_{1+\gamma} + K \|v\|_{1+\gamma'}. \quad (3.5)$$

The proof of Proposition 3.2 is postponed to the end of this subsection as it is based on the auxiliary results presented in Lemma 3.3 and Lemma 3.4 below. We start with an estimate for the commutator  $[\pi_j^\varepsilon, \Phi(\tau u)]$ .

**Lemma 3.3.** *Let  $u \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$  and  $\varepsilon \in (0, 1)$  be given. Then, there is a positive constant  $K = K(\varepsilon)$  such that for all  $\tau \in [0, 1]$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$  we have*

$$\|\pi_j^\varepsilon \Phi(\tau u)[v] - \Phi(\tau u)[\pi_j^\varepsilon v]\|_{\gamma-\alpha} \leq K \|v\|_{1+\gamma-\alpha}. \quad (3.6)$$

*Proof.* In this proof constants denoted by  $C$  depend only on  $u$  and  $\alpha, \beta, \gamma$ , while constants denoted by  $K$  may depend additionally also on  $\varepsilon$ .

Letting  $\varphi := \pi_j^\varepsilon \Phi(\tau u)[v] - \Phi(\tau u)[\pi_j^\varepsilon v]$ , we have

$$\varphi = -\frac{2}{\alpha} (1 + \tau^2 |\nabla u|^2)^{1/2} (\varphi_1 + \varphi_2),$$

where, for  $x \in \mathbb{R}^n$ ,

$$\begin{aligned}\varphi_1(x) &:= \int_{\mathbb{R}^n} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{|y|^{n+1+\alpha}} \frac{y \cdot \nabla v(x-y)}{[1 + \tau^2(\delta_{[x,y]}u/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy, \\ \varphi_2(x) &:= \int_{\mathbb{R}^n} \frac{\delta_{[x,y]}\pi_j^\varepsilon - y \cdot \nabla \pi_j^\varepsilon(x-y)}{|y|^{n+1+\alpha}} \frac{v(x-y)}{[1 + \tau^2(\delta_{[x,y]}u/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy.\end{aligned}$$

Since  $\beta > \gamma - \alpha$  and the little Hölder spaces are Banach algebras, we have

$$\|\varphi\|_{\gamma-\alpha} \leq C(\|\varphi_1\|_{\gamma-\alpha} + \|\varphi_2\|_{\gamma-\alpha}). \quad (3.7)$$

We next estimate  $\|\varphi_1\|_{\gamma-\alpha}$ . Using the mean value theorem, for  $x, y \in \mathbb{R}^n$  with  $y \neq 0$  we have

$$\left| \frac{\delta_{[x,y]}\pi_j^\varepsilon}{|y|^{n+1+\alpha}} \frac{y \cdot \nabla v(x-y)}{[1 + \tau^2(\delta_{[x,y]}u/|y|)^2]^{\frac{n+1+\alpha}{2}}} \right| \leq C \frac{\|v\|_1}{|y|^{n+\alpha}} (\mathbf{1}_{\{|y| \geq 1\}}(y) + \|\pi_j^\varepsilon\|_1 |y| \mathbf{1}_{\{|y| < 1\}}(y)),$$

and therefore

$$\|\varphi_1\|_0 \leq K\|v\|_1. \quad (3.8)$$

In order to estimate the seminorm  $[\varphi_1]_{\gamma-\alpha}$ , let  $x_1, x_2 \in \mathbb{R}^n$  be given. Then, the mean value theorem yields

$$\begin{aligned}|\varphi_1(x_1) - \varphi_1(x_2)| &\leq \int_{\mathbb{R}^n} \frac{|(\delta_{[x_1,y]}\pi_j^\varepsilon)\nabla v(x_1-y) - (\delta_{[x_2,y]}\pi_j^\varepsilon)\nabla v(x_2-y)|}{|y|^{n+\alpha}} dy \\ &\quad + C \int_{\mathbb{R}^n} \frac{|(\delta_{[x_2,y]}\pi_j^\varepsilon)\nabla v(x_2-y)|}{|y|^{n+\alpha}} \left| \frac{\delta_{[x_1,y]}u}{|y|} - \frac{\delta_{[x_2,y]}u}{|y|} \right| dy.\end{aligned}$$

Taking again advantage of the mean value theorem, we obtain

$$\begin{aligned}&|(\delta_{[x_1,y]}\pi_j^\varepsilon)\nabla v(x_1-y) - (\delta_{[x_2,y]}\pi_j^\varepsilon)\nabla v(x_2-y)| \\ &\leq C\|v\|_1 |\delta_{[x_1,y]}\pi_j^\varepsilon - \delta_{[x_2,y]}\pi_j^\varepsilon| + |(\delta_{[x_2,y]}\pi_j^\varepsilon)(\nabla v(x_1-y) - \nabla v(x_2-y))| \\ &\leq C\|v\|_1 (\|\pi_j^\varepsilon\|_{1,\gamma-\alpha} |y| \mathbf{1}_{\{|y| < 1\}}(y) + [\pi_j^\varepsilon]_{\gamma-\alpha} \mathbf{1}_{\{|y| \geq 1\}}(y)) |x_1 - x_2|^{\gamma-\alpha} \\ &\quad + C\|v\|_{1+\gamma-\alpha} (\|\pi_j^\varepsilon\|_1 |y| \mathbf{1}_{\{|y| < 1\}}(y) + \mathbf{1}_{\{|y| \geq 1\}}(y)) |x_1 - x_2|^{\gamma-\alpha}\end{aligned}$$

and, in view of  $\beta > \gamma - \alpha$ ,

$$\begin{aligned}&|(\delta_{[x_2,y]}\pi_j^\varepsilon)\nabla v(x_2-y)(\delta_{[x_2,y]}u - \delta_{[x_1,y]}u)| \\ &\leq C\|v\|_1 [u]_{1,\gamma-\alpha} (\|\pi_j^\varepsilon\|_1 |y|^2 \mathbf{1}_{\{|y| < 1\}}(y) + |y| \mathbf{1}_{\{|y| \geq 1\}}(y)) |x_1 - x_2|^{\gamma-\alpha}.\end{aligned}$$

The latter estimates lead us to

$$[\varphi_1]_{\gamma-\alpha} \leq K\|v\|_{1+\gamma-\alpha}. \quad (3.9)$$

Gathering (3.8)-(3.9), we get

$$\|\varphi_1\|_{\gamma-\alpha} \leq K\|v\|_{1+\gamma-\alpha}. \quad (3.10)$$

Concerning  $\varphi_2$ , we note for  $x \in \mathbb{R}^n$  that

$$\varphi_2(x) = \int_0^1 \int_{\mathbb{R}^n} \frac{y \cdot (\nabla \pi_j^\varepsilon(x-y+sy) - \nabla \pi_j^\varepsilon(x-y))}{|y|^{n+1+\alpha}} \frac{v(x-y)}{[1 + \tau^2(\delta_{[x,y]}u/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy ds,$$

and we therefore may repeat the arguments used to derive (3.10) and conclude that

$$\|\varphi_2\|_{\gamma-\alpha} \leq K\|v\|_{1+\gamma-\alpha}. \quad (3.11)$$

The desired claim follows now from (3.7), (3.10), and (3.11).  $\square$

Lemma 3.4 is the second main ingredient in the proof of Proposition 3.2:

**Lemma 3.4.** *Let  $\gamma' \in (\max\{\alpha, \beta\}, \gamma)$  and let  $u \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$  and  $\nu > 0$  be given. Then, if  $\varepsilon \in (0, 1)$  is sufficiently small, there exists a positive constant  $K = K(\varepsilon)$  such that for all  $\tau \in [0, 1]$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$  we have*

$$\|\Phi(\tau u)[\pi_j^\varepsilon v] - (1 + \tau^2|\nabla u|^2)^{1/2}(x_j^\varepsilon)A^{\tau\nabla u(x_j^\varepsilon)}[\pi_j^\varepsilon v]\|_{\gamma-\alpha} \leq \nu\|\pi_j^\varepsilon v\|_{1+\gamma} + K\|v\|_{1+\gamma'}. \quad (3.12)$$

*Proof.* As before, constants denoted by  $C$  depend only on  $u$  and  $\alpha, \beta, \gamma, \gamma'$ , while constants denoted by  $K$  may depend additionally also on  $\varepsilon$ . Let

$$\psi := \Phi(\tau u)[\pi_j^\varepsilon v] - (1 + \tau^2|\nabla u|^2)^{1/2}(x_j^\varepsilon)A^{\tau\nabla u(x_j^\varepsilon)}[\pi_j^\varepsilon v].$$

Taking into account that  $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$  and recalling (2.3), we have

$$\psi = \psi_1 - \psi_2,$$

where

$$\psi_1 := \chi_j^\varepsilon((1 + \tau^2|\nabla u|^2)^{1/2}A(\tau u)[\pi_j^\varepsilon v] - (1 + \tau^2|\nabla u|^2)^{1/2}(x_j^\varepsilon)A^{\tau\nabla u(x_j^\varepsilon)}[\pi_j^\varepsilon v]),$$

$$\psi_2 := [\chi_j^\varepsilon, \Phi(\tau u)][\pi_j^\varepsilon v] - (1 + \tau^2|\nabla u|^2)^{1/2}(x_j^\varepsilon)[\chi_j^\varepsilon, A^{\tau\nabla u(x_j^\varepsilon)}][\pi_j^\varepsilon v].$$

It is not difficult to see that the arguments in the proof of Lemma 3.3 can also be used to estimate the commutators  $[\chi_j^\varepsilon, \Phi(\tau u)]$  and  $[\chi_j^\varepsilon, A^{\tau\nabla u(x_j^\varepsilon)}]$  which appear in the definition of  $\psi_2$  above. Hence

$$\|\psi_2\|_{\gamma-\alpha} \leq K\|\pi_j^\varepsilon v\|_{1+\gamma-\alpha} \leq K\|v\|_{1+\gamma'} \quad (3.13)$$

as  $\gamma' > \beta > \gamma - \alpha$ . It remains to estimate the function

$$\psi_1(x) = \chi_j^\varepsilon(x) \int_{\mathbb{R}^n} P(x, y) dy, \quad x \in \mathbb{R}^n,$$

where, given  $x, y \in \mathbb{R}^n$  with  $y \neq 0$ , we set

$$P(x, y) := \frac{2}{\alpha} \frac{\delta_{[x,y]}(\pi_j^\varepsilon v) - y \cdot \nabla(\pi_j^\varepsilon v)(x-y)}{|y|^{n+1+\alpha}} \times \left( \frac{(1 + \tau^2|\nabla u|^2)^{1/2}(x)}{[1 + \tau^2(\delta_{[x,y]}u/|y|)^2]^{\frac{n+1+\alpha}{2}}} - \frac{(1 + \tau^2|\nabla u|^2)^{1/2}(x_j^\varepsilon)}{[1 + \tau^2(|y \cdot \nabla u(x_j^\varepsilon)|/|y|)^2]^{\frac{n+1+\alpha}{2}}} \right).$$

We first infer from (2.6) (with  $\gamma$  replaced by  $\gamma'$ ) that

$$\|\psi_1\|_0 \leq C\|\pi_j^\varepsilon v\|_{1+\gamma'} \leq K\|v\|_{1+\gamma'}. \quad (3.14)$$

Let now  $x_1 \neq x_2 \in \mathbb{R}^n$ . In order to estimate the difference  $|\psi_1(x_1) - \psi_1(x_2)|$  we may assume, in view of the property  $\text{supp } \chi_j^\varepsilon \subset \overline{\mathbb{B}_{3\varepsilon}(x_j^\varepsilon)} + 2\pi\mathbb{Z}^n$ , that  $x_1, x_2 \in \overline{\mathbb{B}_{3\varepsilon}(x_j^\varepsilon)}$ . We then have

$$|\psi(x_1) - \psi(x_2)| \leq S_1 + S_2,$$

where

$$S_1 := \int_{\mathbb{R}^n} |P(x_1, y) - P(x_2, y)| dy \quad \text{and} \quad S_2 := |\chi_j^\varepsilon(x_1) - \chi_j^\varepsilon(x_2)| \int_{\mathbb{R}^n} |P(x_2, y)| dy.$$

The estimate (2.6) (with  $\gamma$  replaced by  $\gamma'$ ) yields

$$S_2 \leq C[\chi_j^\varepsilon]_{\gamma-\alpha} \|\pi_j^\varepsilon v\|_{1+\gamma'} |x_1 - x_2|^{\gamma-\alpha} \leq K \|v\|_{1+\gamma'} |x_1 - x_2|^{\gamma-\alpha}, \quad (3.15)$$

and it remains to estimate the term  $S_1$ . To this end we use the mean value theorem to derive

$$|P(x_1, y) - P(x_2, y)| \leq C(L_1(y) + L_2(y) + L_3(y)), \quad y \in \mathbb{R}^n \setminus \{0\},$$

where

$$\begin{aligned} L_1(y) &:= \frac{|\delta_{[x_1, y]}(\pi_j^\varepsilon v) - y \cdot \nabla(\pi_j^\varepsilon v)(x_1 - y) - [\delta_{[x_2, y]}(\pi_j^\varepsilon v) - y \cdot \nabla(\pi_j^\varepsilon v)(x_2 - y)]|}{|y|^{n+1+\alpha}} \\ &\quad \times |\nabla u(x_1) - \nabla u(x_2^\varepsilon)|, \\ L_2(y) &:= \frac{|\delta_{[x_1, y]}(\pi_j^\varepsilon v) - y \cdot \nabla(\pi_j^\varepsilon v)(x_1 - y) - [\delta_{[x_2, y]}(\pi_j^\varepsilon v) - y \cdot \nabla(\pi_j^\varepsilon v)(x_2 - y)]|}{|y|^{n+1+\alpha}} \\ &\quad \times \left| \frac{\delta_{[x_1, y]} u}{|y|} - \frac{y \cdot \nabla u(x_2^\varepsilon)}{|y|} \right|, \\ L_3(y) &:= \frac{|\delta_{[x_2, y]}(\pi_j^\varepsilon v) - y \cdot \nabla(\pi_j^\varepsilon v)(x_2 - y)|}{|y|^{n+1+\alpha}} \left( |\nabla u(x_1) - \nabla u(x_2)| + \left| \frac{\delta_{[x_1, y]} u}{|y|} - \frac{\delta_{[x_2, y]} u}{|y|} \right| \right). \end{aligned}$$

Recalling (2.8), we have in view of  $u \in \mathbf{h}^{1+\beta}(\mathbb{T}^n)$  and  $x_1 \in \overline{\mathbb{B}_{3\varepsilon}(x_j^\varepsilon)}$

$$L_1(y) \leq C\varepsilon^\beta \|\pi_j^\varepsilon v\|_{1+\gamma} \left( \frac{|x_1 - x_2|^\gamma}{|y|^{n+\alpha}} \mathbf{1}_{\{|y| \geq |x_1 - x_2|\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma}} \mathbf{1}_{\{|y| < |x_1 - x_2|\}}(y) \right).$$

Hence, if  $\varepsilon$  is sufficiently small, then

$$\int_{\mathbb{R}^n} CL_1(y) dy \leq \frac{\nu}{2} \|\pi_j^\varepsilon v\|_{1+\gamma} |x_1 - x_2|^{\gamma-\alpha} \quad (3.16)$$

for all  $\tau \in [0, 1]$ ,  $v \in \mathbf{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$ . Moreover, since  $|x_1 - x_2| \leq 6\varepsilon$ , the estimate (2.8) and the mean value theorem lead us to

$$\begin{aligned} L_2(y) &\leq C\varepsilon^\beta \|\pi_j^\varepsilon v\|_{1+\gamma} \left( \frac{|x_1 - x_2|^\gamma}{|y|^{n+\alpha}} \mathbf{1}_{\{|x_1 - x_2| \leq |y|\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma}} \mathbf{1}_{\{|y| < |x_1 - x_2|\}}(y) \right) \mathbf{1}_{\{|y| \leq 6\varepsilon\}}(y) \\ &\quad + C \frac{1}{|y|^{n+\alpha}} [\pi_j^\varepsilon v]_{1, \gamma-\alpha} |x_1 - x_2|^{\gamma-\alpha} \mathbf{1}_{\{|y| > 6\varepsilon\}}(y) \\ &\leq C\varepsilon^\beta \|\pi_j^\varepsilon v\|_{1+\gamma} \left( \frac{|x_1 - x_2|^\gamma}{|y|^{n+\alpha}} \mathbf{1}_{\{|x_1 - x_2| \leq |y|\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma}} \mathbf{1}_{\{|y| < |x_1 - x_2|\}}(y) \right) \\ &\quad + K \frac{1}{|y|^{n+\alpha}} \|v\|_{1+\gamma'} |x_1 - x_2|^{\gamma-\alpha} \mathbf{1}_{\{|y| > 6\varepsilon\}}(y). \end{aligned}$$

Therefore, if  $\varepsilon$  is sufficiently small, then

$$\int_{\mathbb{R}^n} CL_2(y) dy \leq \left( \frac{\nu}{2} \|\pi_j^\varepsilon v\|_{1+\gamma} + K \|v\|_{1+\gamma'} \right) |x_1 - x_2|^{\gamma-\alpha} \quad (3.17)$$

for all  $\tau \in [0, 1]$ ,  $v \in \mathbf{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$ .

Finally, (2.6) (with  $\gamma = \gamma'$ ), the inequality  $\beta > \gamma - \alpha$ , and the mean value theorem imply that

$$L_3(y) \leq C \|\pi_j^\varepsilon v\|_{1+\gamma'} \left( \frac{1}{|y|^{n+\alpha}} \mathbf{1}_{\{|y| \geq 1\}}(y) + \frac{1}{|y|^{n+\alpha-\gamma'}} \mathbf{1}_{\{|y| < 1\}}(y) \right) |x_1 - x_2|^{\gamma-\alpha},$$

hence

$$\int_{\mathbb{R}^n} CL_3(y) dy \leq K \|v\|_{1+\gamma'} |x_1 - x_2|^{\gamma-\alpha} \quad (3.18)$$

for all  $\tau \in [0, 1]$ ,  $v \in \mathbf{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$ .

The estimate (3.12) follows now from (3.13)-(3.18).  $\square$

It remains to note that Proposition 3.2 is established:

*Proof of Proposition 3.2.* The assertion follows from Lemma 3.3 and Lemma 3.4.  $\square$

**3.2. Estimates for the Fourier Multipliers.** In Proposition 3.2 we have locally approximated the operator  $\Phi(\tau u)$  with  $\tau \in [0, 1]$  and  $u \in \mathbf{h}^{1+\beta}(\mathbb{T}^n)$  by operators  $\delta A^a$  defined in (3.1) with  $\delta \in [1, \eta]$  and  $a \in \mathbb{R}^n$  satisfying  $|a| \leq \eta$ , where

$$\eta := \|(1 + |\nabla u|^2)^{1/2}\|_0 \geq 1. \quad (3.19)$$

We first show that these operators are Fourier multipliers and then establish in Proposition 3.5 a fundamental estimate for their resolvents.

To start with, given  $a \in \mathbb{R}^n$  and  $k \in \mathbb{Z}^n$ , we use Fubini's theorem to compute

$$\begin{aligned}
& \int_{[-\pi, \pi]^n} \left( \int_{\mathbb{R}^n} \frac{\delta_{[x, y]} v - y \cdot \nabla v(x - y)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \right) e^{-ik \cdot x} dx \\
&= \int_{\mathbb{R}^n} \frac{1}{|y|^{n+1+\alpha}} \left( \int_{[-\pi, \pi]^n} (\delta_{[x, y]} v - y \cdot \nabla v(x - y)) e^{-ik \cdot x} dx \right) \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \\
&= (2\pi)^n \widehat{v}(k) \int_{\mathbb{R}^n} \frac{1 - e^{-iy \cdot k} - iy \cdot k e^{-iy \cdot k}}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \\
&= (2\pi)^n \widehat{v}(k) \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot k) - y \cdot k \sin(y \cdot k)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy, \tag{3.20}
\end{aligned}$$

where we used that

$$\int_{\mathbb{R}^n} \frac{\sin(y \cdot k) - y \cdot k \cos(y \cdot k)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy = 0$$

since the integrand is an odd function. The Fourier coefficients  $\widehat{v}(k)$ ,  $k \in \mathbb{Z}^n$ , of the periodic function  $v$  are defined in Appendix A. Taking into account that  $\nabla(1 - \cos(y \cdot k)) = k \sin(y \cdot k)$ , an application of Gauss' theorem leads us to

$$\begin{aligned}
& \int_{\mathbb{R}^n} \frac{y \cdot k \sin(y \cdot k)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \\
&= - \int_{\mathbb{R}^n} (1 - \cos(y \cdot k)) \operatorname{div} \left( \frac{y}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} \right) dy \\
&= - \int_{\mathbb{R}^n} (1 - \cos(y \cdot k)) \operatorname{div} \left( \frac{y}{|y|^{n+1+\alpha}} \right) \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \\
&= (1 + \alpha) \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot k)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy, \tag{3.21}
\end{aligned}$$

since

$$y \cdot \nabla \left( \frac{|y \cdot a|}{|y|} \right)^2 = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Consequently, we derive from (3.1), (3.20), and (3.21) that

$$\widehat{A^a[v]}(k) = m_a(k) \widehat{v}(k), \quad k \in \mathbb{Z}^n; \tag{3.22}$$

that is,  $A^a$  defined in (3.1) is for  $a \in \mathbb{R}^n$  indeed a Fourier multiplier and its symbol is given by

$$m_a(k) := -2 \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot k)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy, \quad k \in \mathbb{Z}^n. \quad (3.23)$$

Our next purpose is to establish the following fundamental resolvent estimate for the Fourier multiplier  $A^a$ :

**Proposition 3.5.** *Given  $\eta \geq 1$ , there exists a constant  $\kappa = \kappa(\eta) \geq 1$  such that*

$$\kappa \|(\lambda - \delta A^a)[v]\|_{\gamma-\alpha} \geq |\lambda| \|v\|_{\gamma-\alpha} + \|v\|_{1+\gamma} \quad (3.24)$$

for all  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ ,  $\operatorname{Re} \lambda \geq 1$ , and  $v \in \mathbf{h}^{1+\gamma}(\mathbb{T}^n)$ .

Proposition 3.5 will be a consequence of Lemma 3.7 and Lemma 3.8. As a preliminary step, we extend the symbol  $m_a$  defined in (3.23) to the whole  $\mathbb{R}^n$  by setting

$$m_a(x) := -2 \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot x)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy, \quad x \in \mathbb{R}^n. \quad (3.25)$$

In Lemma 3.6 we now show that  $m_a$  is smooth on  $\mathbb{R}^n \setminus \{0\}$  and in Lemma 3.7 we then establish some estimates related to  $m_a$  which are used in the proof of (3.24). Note that it is not obvious at first sight that  $m_a$  is smooth in  $\mathbb{R}^n \setminus \{0\}$ . If  $a = 0$ , this function coincides, up to a multiplicative negative constant, with  $|x|^{1+\alpha}$ , see the proof of Lemma 3.6 below. However, if  $a \neq 0$ , it is not clear whether a similar explicit formula can be derived for  $m_a$ .

**Lemma 3.6.** *Given  $a \in \mathbb{R}^n$ , the function  $m_a : \mathbb{R}^n \rightarrow \mathbb{R}$  defined in (3.25) is smooth in  $\mathbb{R}^n \setminus \{0\}$ .*

*Proof.* Let  $\{e_1, \dots, e_n\}$  be the canonical basis of  $\mathbb{R}^n$ . In order to prove that  $m_a$  is smooth in  $\mathbb{R}^n \setminus (e_n[0, \infty))$ , we denote for each given  $x \in \mathbb{R}^n \setminus (e_n[0, \infty))$  by  $\mathbf{H}_n = \mathbf{H}_n(x) \in \mathbb{R}^{n \times n}$  the Householder transformation

$$\mathbf{H}_n := I_n - 2ww^\top,$$

where

$$w := w(x) := \frac{x/|x| - e_n}{|x/|x| - e_n|} \in \mathbb{R}^n$$

and  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix. Then  $\mathbf{H}_n$  is symmetric, orthogonal, and  $\mathbf{H}_n x = |x|e_n$ . Let further

$$f_n : V_n := (0, \infty) \times (0, 2\pi) \times (0, \pi)^{n-2} \rightarrow \mathbb{R}^n,$$

with  $f_n = f_n(r, \varphi, \vartheta) = r g_n(\varphi, \vartheta)$ ,  $\vartheta = (\vartheta_1, \dots, \vartheta_{n-2})$ , and  $|g_n(\varphi, \vartheta)| = 1$  be the standard  $n$ -dimensional polar coordinates transformation, see e.g [3]. Write  $g_n = (g_n^1, \dots, g_n^n)$ . Then, changing variables according to  $y = \mathbf{H}_n f_n(r, \varphi, \vartheta)$  and using the orthogonality and symmetry

of  $\mathbf{H}_n$ , we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot x)}{|y|^{n+1+\alpha}} \frac{1}{[1 + (|y \cdot a|/|y|)^2]^{\frac{n+1+\alpha}{2}}} dy \\
&= \int_{V_n} \frac{1 - \cos(f_n(r, \varphi, \vartheta) \cdot \mathbf{H}_n x)}{|f_n(r, \varphi, \vartheta)|^{n+1+\alpha}} \frac{|\det(\partial f_n(r, \varphi, \vartheta))|}{[1 + (|f_n(r, \varphi, \vartheta) \cdot \mathbf{H}_n a|/|f_n(r, \varphi, \vartheta)|)^2]^{\frac{n+1+\alpha}{2}}} d(r, \varphi, \vartheta) \\
&= \int_{V_n} \frac{1 - \cos(|x| r g_n^n(\varphi, \vartheta))}{r^{2+\alpha}} \frac{\sin(\vartheta_1) \sin^2(\vartheta_2) \dots \sin^{n-2}(\vartheta_{n-2})}{[1 + |g_n(\varphi, \vartheta) \cdot \mathbf{H}_n a|^2]^{\frac{n+1+\alpha}{2}}} d(r, \varphi, \vartheta) \\
&= |x|^{1+\alpha} \int_{V_n} \frac{1 - \cos(\tau g_n^n(\varphi, \vartheta))}{\tau^{2+\alpha}} \frac{\sin(\vartheta_1) \sin^2(\vartheta_2) \dots \sin^{n-2}(\vartheta_{n-2})}{[1 + |g_n(\varphi, \vartheta) \cdot \mathbf{H}_n a|^2]^{\frac{n+1+\alpha}{2}}} d(\tau, \varphi, \vartheta),
\end{aligned}$$

hence

$$m_a(x) = |x|^{1+\alpha} p_n(x), \quad x \in \mathbb{R}^n \setminus (e_n[0, \infty)),$$

where  $p_n : \mathbb{R}^n \setminus (e_n[0, \infty)) \rightarrow (-\infty, 0)$  is defined by the relation

$$p_n(x) := -2 \int_{V_n} \frac{1 - \cos(\tau g_n^n(\varphi, \vartheta))}{\tau^{2+\alpha}} \frac{\sin(\vartheta_1) \sin^2(\vartheta_2) \dots \sin^{n-2}(\vartheta_{n-2})}{[1 + |\mathbf{H}_n(x) a \cdot g_n(\varphi, \vartheta)|^2]^{\frac{n+1+\alpha}{2}}} d(\tau, \varphi, \vartheta) \quad (3.26)$$

with  $\mathbf{H}_n = \mathbf{H}_n(x) = (h_{ij}(x))_{1 \leq i, j \leq n}$  and

$$h_{ij}(x) := \delta_{ij} - \left( \frac{x_i}{|x|} - \delta_{in} \right) \left( \frac{x_j}{|x|} - \delta_{jn} \right) \left( 1 - \frac{x_n}{|x|} \right)^{-1}. \quad (3.27)$$

Clearly, all functions  $h_{ij}$ ,  $1 \leq i, j \leq n$ , are smooth in  $\mathbb{R}^n \setminus (e_n[0, \infty))$  and their partial derivatives are all bounded on compact subsets of  $\mathbb{R}^n \setminus (e_n[0, \infty))$ . A direct application of the theorem on differentiation under the integral sign allows one to conclude that  $p_n$ , and hence also  $m_a$ , is smooth in  $\mathbb{R}^n \setminus (e_n[0, \infty))$ .

Arguing similarly as above, but using instead of  $\mathbf{H}_n$  the Householder transformation  $\mathbf{H}_1 = \mathbf{H}_1(x)$  which maps  $x \in \mathbb{R}^n \setminus (e_1[0, \infty))$  to  $|x|e_1$ , we may deduce that  $m_a$  is smooth also in  $\mathbb{R}^n \setminus (e_1[0, \infty))$  and the desired claim follows.  $\square$

In view of Lemma 3.6, the function  $P_a : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ , defined for fixed  $a \in \mathbb{R}^n$  by

$$P_a(x) := \frac{m_a(x)}{|x|^{1+\alpha}}, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (3.28)$$

is smooth, and we infer from the definition (3.25) that  $P_a(x) = P_a(-x)$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ . Lemma 3.7 below provides some useful estimates on  $P_a$ . Before stating this result, we note that if  $\mathcal{O} \subset \mathbb{R}^n$  is an open set and  $f : \mathcal{O} \rightarrow \mathbb{R}$  is a positive function, then for each  $0 \neq \mu \in \mathbb{N}^n$ , we have

$$\partial^\mu \left( \frac{1}{f} \right) = \sum_{k=1}^{|\mu|} \frac{1}{f^{k+1}} \sum_{\substack{\{\beta_1, \dots, \beta_k\} \subset \mathbb{N}^n \\ \beta_1 + \dots + \beta_k = \mu}} c_{\beta_1 \dots \beta_k} \partial^{\beta_1} f \dots \partial^{\beta_k} f, \quad (3.29)$$

with constants  $c_{\beta_1 \dots \beta_k} \in \mathbb{Z}$  independent of  $f$ . The formula (3.29) is obtained by using a standard induction argument. In Lemma 3.7-Lemma 3.9 below we use the notation

$$N := [n/2] + 1, \quad (3.30)$$

where  $[n/2]$  is the integer part of  $n/2$ .

**Lemma 3.7.** *Let  $P_a$  be defined in (3.28). Given  $\eta \geq 1$ , there exists a constant  $M = M(\eta) \geq 1$  such that for all  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$  we have*

$$M^{-1} \leq \inf_{x \in \mathbb{R}^n \setminus \{0\}} |P_a(x)| \quad \text{and} \quad \sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{|\mu|} |\partial^\mu P_a(x)| \leq M, \quad (3.31)$$

whenever  $\mu \in \mathbb{N}^n$  satisfies  $|\mu| \leq N$ .

*Proof.* Since  $P_a$  is smooth and  $P_a(x) = P_a(-x)$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ , it suffices to estimate  $P_a$  and its partial derivatives in the set  $\{x_n < 0\}$ , in which case we also have  $P_a = p_n$ , see (3.26). It readily follows from (3.26) that, if  $x \in \mathbb{R}^n \setminus (e_n[0, \infty))$ , then

$$|p_n(x)| \geq \frac{2}{[1 + \eta^2]^{\frac{n+1+\alpha}{2}}} \int_{V_n} \frac{1 - \cos(\tau g_n^n(\varphi, \vartheta))}{\tau^{2+\alpha}} \sin(\vartheta_1) \sin^2(\vartheta_2) \dots \sin^{n-2}(\vartheta_{n-2}) d(\tau, \varphi, \vartheta).$$

This proves the first estimate in (3.31).

Observing for  $x \in \{x_n < 0\}$  that

$$1 - \frac{x_n}{|x|} \geq 1,$$

it follows that the functions  $h_{ij}$ ,  $1 \leq i, j \leq n$ , defined in (3.27) are smooth in  $\{x_n < 0\}$  and

$$\max_{|\mu| \leq N} \sup_{\{x_n < 0\}} |x|^{|\mu|} |\partial^\mu h_{ij}(x)| < \infty \quad \text{for all } 1 \leq i, j \leq n.$$

A direct application of the theorem on the differentiation under the integral sign and of (3.29) enables us to deduce also the second estimate in (3.31).  $\square$

With these preparatory results we may now establish the resolvent estimate on  $\delta A^a$  stated in Proposition 3.5. For this purpose, let  $\eta \geq 1$  be as in (3.19) and consider  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ , and  $\text{Re } \lambda \geq 1$ . We introduce  $R(\lambda)$  as the Fourier multiplier

$$\widehat{R(\lambda)[v]}(k) := \frac{\mathbf{m}_{1+\alpha}(k)}{\lambda - \delta m_a(k)} \widehat{v}(k), \quad k \in \mathbb{Z}^n, \quad (3.32)$$

with  $m_a$  being the Fourier symbol of  $A^a$  defined in (3.25) and  $\mathbf{m}_{1+\alpha}$  is a smooth function defined on  $\mathbb{R}^n$  such that  $\mathbf{m}_{1+\alpha}(x) = 1$  for  $|x| \leq 1/2$  and  $\mathbf{m}_{1+\alpha}(x) = |x|^{1+\alpha}$  for  $|x| \geq 1$ . Using the Mihlin-Hörmander multiplier theorem we shall show below that  $R(\lambda) \in \mathcal{L}(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$  and that there exists  $\kappa = \kappa(\eta)$  such that

$$\|R(\lambda)\|_{\mathcal{L}(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))} \leq \kappa, \quad (3.33)$$

whenever  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ , and  $\text{Re } \lambda \geq 1$ . We also note that the operator  $I_{-1-\alpha}$  from Lemma A.1 with Fourier symbol  $1/\mathbf{m}_{1+\alpha}$  satisfies

$$I_{-1-\alpha} \in \mathcal{L}(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n), \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)). \quad (3.34)$$

Indeed, (3.34) readily follows from Lemma A.1, the fact that one may identify  $C^s(\mathbb{T}^n)$  with the Besov space  $B_{\infty,\infty}^s(\mathbb{T}^n)$  for  $s > 0$  with  $s \notin \mathbb{N}$ , e.g. see [37], and the fact that  $I_{-1-\alpha}$  maps  $C^\infty(\mathbb{T}^n)$  to  $C^\infty(\mathbb{T}^n)$ . Therefore, we infer from (3.33) and (3.34) that

$$(\lambda - \delta A^a)^{-1} = I_{-1-\alpha} R(\lambda) \in \mathcal{L}(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n), \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)). \quad (3.35)$$

This is the content of the following lemma.

**Lemma 3.8.** *Given  $\eta \geq 1$ , there exists a constant  $\kappa = \kappa(\eta) \geq 1$  such that if  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ ,  $\operatorname{Re} \lambda \geq 1$ , and  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , then*

$$\kappa \|(\lambda - \delta A^a)[v]\|_{\gamma-\alpha} \geq \|v\|_{1+\gamma}. \quad (3.36)$$

Moreover,  $\lambda - \delta A^a \in \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$  is an isomorphism.

*Proof.* As pointed out above, (3.36) follows from (3.33), (3.34), and (3.35). It thus remains to establish (3.33), for which the Mihlin-Hörmander multiplier theorem [37, Section 3.6.3/Remark 3] entails that it suffices to show that

$$\sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{|\beta|} \left| \partial^\beta \left( \frac{|x|^{1+\alpha}}{\lambda - \delta m_a(x)} \right) \right| \leq \kappa, \quad (3.37)$$

uniformly with respect to  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ , and  $\operatorname{Re} \lambda \geq 1$ , with  $N$  given by (3.30). Let thus  $\beta \in \mathbb{N}^n$  satisfy  $|\beta| \leq N$ . In view of Leibniz' rule, it remains to estimate terms of the form

$$|x|^{|\beta|} |\partial^{\beta-\mu} |x|^{1+\alpha}| \left| \partial^\mu \left( \frac{1}{\lambda - \delta m_a(x)} \right) \right|$$

with  $\mu \in \mathbb{N}^n$  such that  $\mu \leq \beta$ . To this end, first note that

$$|\partial^{\beta-\mu} |x|^{1+\alpha}| \leq C |x|^{1+\alpha+|\mu|-|\beta|}, \quad (3.38)$$

where  $C = C(N)$ . Moreover, it follows from (3.29) that

$$\partial^\mu ((\lambda - \delta m_a)^{-1}) = \sum_{k=1}^{|\mu|} \frac{\delta^k}{(\lambda - \delta m_a)^{k+1}} \sum_{\substack{\{\beta_1, \dots, \beta_k\} \subset \mathbb{N}^n \\ \beta_1 + \dots + \beta_k = \mu}} c_{\beta_1 \dots \beta_k} \partial^{\beta_1} m_a \cdot \dots \cdot \partial^{\beta_k} m_a, \quad (3.39)$$

with constants  $c_{\beta_1 \dots \beta_k} \in \mathbb{Z}$  independent of  $\lambda$ ,  $\delta$ , and  $m_a$ . We note that  $m_a(x) < 0$  for  $x \neq 0$ . Therefore, given  $\delta \in [1, \eta]$ ,  $\operatorname{Re} \lambda \geq 1$ , and  $x \in \mathbb{R}^n \setminus \{0\}$ , we infer from (3.28) and Lemma 3.7 that

$$\begin{aligned} |\lambda - \delta m_a(x)| &= \sqrt{(\operatorname{Re} \lambda - \delta m_a(x))^2 + (\operatorname{Im} \lambda)^2} \geq \max\{|\lambda|, |m_a(x)|\} \\ &\geq \max\{|\lambda|, M^{-1}|x|^{1+\alpha}\}, \end{aligned} \quad (3.40)$$

with  $M = M(\eta) \geq 1$ , and therefore

$$\frac{1}{|\lambda - \delta m_a(x)|^{k+1}} \leq \frac{M^{k+1}}{|x|^{(k+1)(1+\alpha)}}, \quad k \geq 0. \quad (3.41)$$

Leibniz' rule, (3.38), and Lemma 3.7 ensure the existence of a constant  $C = C(\eta)$  such that for all  $x \in \mathbb{R}^n \setminus \{0\}$  we have

$$|(\partial^{\beta_1} m_a \cdot \dots \cdot \partial^{\beta_k} m_a)(x)| \leq C |x|^{1+\alpha-|\beta_1|} \dots |x|^{1+\alpha-|\beta_k|} = C |x|^{k(1+\alpha)-|\mu|}. \quad (3.42)$$

The estimate (3.37) is now a straightforward consequence of (3.38), (3.41), and (3.42).  $\square$

To complete the proof of Proposition 3.5, we establish the following resolvent estimate:

**Lemma 3.9.** *Given  $\eta \geq 1$ , there exists a constant  $\kappa = \kappa(\eta) \geq 1$  such that*

$$\|(\lambda - \delta A^a)^{-1}\|_{\mathcal{L}(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))} \leq \frac{\kappa}{|\lambda|} \quad (3.43)$$

for all  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ , and  $\operatorname{Re} \lambda \geq 1$ .

*Proof.* Since  $(\lambda - \delta A^a)^{-1} = I_{-1-\alpha} R(\lambda)$ , the Mikhlin-Hörmander multiplier theorem [37, Section 3.6.3/Remark 3] implies that we only have to show that

$$\sup_{|\mu| \leq N} \sup_{x \in \mathbb{R}^n \setminus \{0\}} |x|^{|\mu|} \left| \partial^\mu ((\lambda - \delta m_a)^{-1})(x) \right| \leq \frac{\kappa}{|\lambda|} \quad (3.44)$$

uniformly in  $\delta \in [1, \eta]$ ,  $a \in \mathbb{R}^n$  with  $|a| \leq \eta$ , and  $\operatorname{Re} \lambda \geq 1$ .

Invoking (3.40), there exists a constant  $M = M(\eta) \geq 1$  such that for all  $\delta \in [1, \eta]$ ,  $\operatorname{Re} \lambda \geq 1$ , and  $x \in \mathbb{R}^n \setminus \{0\}$ , we have

$$\frac{1}{|\lambda - \delta m_a(x)|^{k+1}} \leq \frac{1}{|\lambda|} \frac{M^k}{|x|^{k(1+\alpha)}}, \quad k \geq 0.$$

The desired claim (3.44) follows now from the latter estimate, (3.39), and (3.42).  $\square$

*Proof of Proposition 3.5.* This is now a consequence of Lemma 3.7 and Lemma 3.9.  $\square$

**3.3. Proof of Theorem 3.1.** Having established Proposition 3.2 and Proposition 3.5, we turn to the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Let  $\gamma' \in (\max\{\alpha, \beta\}, \gamma)$ , let  $\eta \geq 1$  be as defined in (3.19), and let  $\kappa \geq 1$  be chosen as in Proposition 3.5. Defining  $\nu := (2\kappa)^{-1}$ , Proposition 3.2 ensures there exist  $\varepsilon \in (0, 1)$  and a positive constant  $K = K(\varepsilon)$  with the property that for all  $\tau \in [0, 1]$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $0 \leq j \leq N$  we have

$$2\kappa \left\| \pi_j^\varepsilon \Phi(\tau u)[v] - (1 + \tau^2 |\nabla u|^2)^{1/2} (x_j^\varepsilon) A^{\tau \nabla u(x_j^\varepsilon)} [\pi_j^\varepsilon v] \right\|_{\gamma-\alpha} \leq \|\pi_j^\varepsilon v\|_{1+\gamma} + 2\kappa K \|v\|_{1+\gamma'}. \quad (3.45)$$

We now infer from Proposition 3.5 that for all  $\tau \in [0, 1]$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ ,  $0 \leq j \leq N$ , and  $\operatorname{Re} \lambda \geq 1$

$$2\kappa \left\| (\lambda - (1 + \tau^2 |\nabla u|^2)^{1/2} (x_j^\varepsilon) A^{\tau \nabla u(x_j^\varepsilon)}) [\pi_j^\varepsilon v] \right\|_{\gamma-\alpha} \geq 2|\lambda| \|\pi_j^\varepsilon v\|_{\gamma-\alpha} + 2\|\pi_j^\varepsilon v\|_{1+\gamma}. \quad (3.46)$$

Combining (3.45) and (3.46), we get

$$\begin{aligned} 2\kappa \left\| \pi_j^\varepsilon (\lambda - \Phi(\tau u))[v] \right\|_{\gamma-\alpha} &\geq 2\kappa \left\| (\lambda - (1 + \tau^2 |\nabla u|^2)^{1/2} (x_j^\varepsilon) A^{\tau \nabla u(x_j^\varepsilon)}) [\pi_j^\varepsilon v] \right\|_{\gamma-\alpha} \\ &\quad - 2\kappa \left\| \pi_j^\varepsilon \Phi(\tau u)[v] - (1 + \tau^2 |\nabla u|^2)^{1/2} (x_j^\varepsilon) A^{\tau \nabla u(x_j^\varepsilon)} [\pi_j^\varepsilon v] \right\|_{\gamma-\alpha} \\ &\geq 2|\lambda| \|\pi_j^\varepsilon v\|_{\gamma-\alpha} + \|\pi_j^\varepsilon v\|_{1+\gamma} - 2\kappa K \|v\|_{1+\gamma'}. \end{aligned}$$

Summing over  $j$ , we deduce from (3.4), Young's inequality, and the interpolation property (2.1) that there exist constants  $\kappa_1 \geq 1$  and  $\omega_1 \geq 1$  such that

$$\kappa_1 \left\| (\lambda - \Phi(\tau u))[v] \right\|_{\gamma-\alpha} \geq |\lambda| \|v\|_{\gamma-\alpha} + \|v\|_{1+\gamma} \quad (3.47)$$

for all  $\tau \in [0, 1]$ ,  $v \in \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)$ , and  $\operatorname{Re} \lambda \geq \omega_1$ .

Moreover, Lemma 3.8 ensures that

$$\omega_1 - \Phi(0) = \omega_1 - A^0 \in \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$$

is an isomorphism. This property, the method of continuity [2, Proposition I.1.1.1], and (3.47) imply now that also  $\omega_1 - \Phi(u) \in \mathcal{L}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$  is an isomorphism. Consequently, together with (3.47) (with  $\tau = 1$ ) we infer from [2, Chapter I] that indeed

$$-\Phi(u) \in \mathcal{H}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))$$

as claimed.  $\square$

**3.4. Proof of Theorem 1.1.** We are now in a position to establish our first main result stated in Theorem 1.1.

*Proof of Theorem 1.1.* Let  $\beta' \in (0, \beta)$  be chosen such that  $\gamma < \alpha + \beta'$ . Combining Proposition 2.1 and Theorem 3.1 we have

$$-\Phi \in C^\infty(\mathfrak{h}^{1+\beta'}(\mathbb{T}^n), \mathcal{H}(\mathfrak{h}^{1+\gamma}(\mathbb{T}^n), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n))). \quad (3.48)$$

Moreover, setting

$$\theta' := 1 - \frac{\gamma - \beta'}{1 + \alpha} \quad \text{and} \quad \theta := 1 - \frac{\gamma - \beta}{1 + \alpha},$$

we infer from (2.1) that

$$(\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n), \mathfrak{h}^{1+\gamma}(\mathbb{T}^n))_{\theta', \infty}^0 = \mathfrak{h}^{1+\beta'}(\mathbb{T}^n) \quad \text{and} \quad (\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n), \mathfrak{h}^{1+\gamma}(\mathbb{T}^n))_{\theta, \infty}^0 = \mathfrak{h}^{1+\beta}(\mathbb{T}^n).$$

The property (3.48) and the latter interpolation relations ensure that we may apply [33, Theorem 1.1] to the evolution problem (1.4). Hence, given  $u_0 \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$ , there exists a unique maximal classical solution  $u = u(\cdot; u_0)$  to (1.4) such that

$$u \in C([0, T^+), \mathfrak{h}^{1+\beta}(\mathbb{T}^n)) \cap C((0, T^+), \mathfrak{h}^{1+\gamma}(\mathbb{T}^n)) \cap C^1((0, T^+), \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n)) \quad (3.49)$$

and

$$u \in C^{\theta-\theta'}([0, T^+), \mathfrak{h}^{1+\beta'}(\mathbb{T}^n)),$$

where  $T^+ = T^+(u_0) \in (0, \infty]$  denotes the maximal existence time. Moreover, the mapping  $[(t, u_0) \mapsto u(t; u_0)]$  defines a semiflow on  $\mathfrak{h}^{1+\beta}(\mathbb{T}^n)$  which is smooth in the open set

$$\{(t, u_0) : u_0 \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n), 0 < t < T^+(u_0)\} \subset \mathbb{R} \times \mathfrak{h}^{1+\beta}(\mathbb{T}^n).$$

As shown in Lemma 4.2 below, the solution is actually unique in the larger set of functions satisfying only (3.49).

Since the embedding  $\mathfrak{h}^{1+\gamma}(\mathbb{T}^n) \hookrightarrow \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n)$  is compact, Theorem 1.1 (i) follows directly from [33, Theorem 1.1 (iv)( $\beta$ )].

We next establish the parabolic smoothing property (1.7). This may be shown by using a parameter trick employed also in other settings, see [4, 25, 35]. The arguments are more or less identical to those presented in [32, Theorem 1.3] and we include them here merely for the reader's ease.

To start with, we fix a maximal solution  $u = u(\cdot; u_0)$  to (1.4). Given  $\lambda_1 \in (0, \infty)$  and  $\lambda_2 \in \mathbb{R}^n$ , we set  $\lambda := (\lambda_1, \lambda_2)$  and define

$$u_\lambda(t, x) := u(\lambda_1 t, x + \lambda_2 t), \quad x \in \mathbb{R}^n, \quad 0 \leq t < T^+(u_0)/\lambda_1.$$

Using the density of  $C^\infty(\mathbb{T}^n)$  in the little Hölder spaces, it is possible to show that

$$u_\lambda \in C([0, T_\lambda^+), h^{1+\beta}(\mathbb{T}^n)) \cap C((0, T_\lambda^+), h^{1+\gamma}(\mathbb{T}^n)) \cap C^1((0, T_\lambda^+, h^{\gamma-\alpha}(\mathbb{T}^n))), \quad (3.50)$$

where  $T_\lambda^+ := T^+(u_0)/\lambda_1$ , and

$$\frac{du_\lambda}{dt}(t) = \lambda_1 \Phi(u_\lambda(t))[u_\lambda(t)] + \lambda_2 \cdot \nabla u_\lambda(t), \quad t \in (0, T_\lambda^+). \quad (3.51)$$

Let now  $U := (U_1, U_2, U_3) : [0, T_\lambda^+) \rightarrow \mathbb{R}^{n+1} \times h^{1+\beta}(\mathbb{T}^n)$  be the function defined by

$$(U_1, U_2, U_3)(t) := (\lambda_1, \lambda_2, u_\lambda(t)).$$

Then, in view of (3.51), the function  $U$  is a solution to the quasilinear evolution problem

$$\frac{dU}{dt}(t) = \Psi(U(t))[U(t)], \quad t > 0, \quad U(0) = U_0 := (\lambda, u_0), \quad (3.52)$$

where

$$\Psi : (0, \infty) \times \mathbb{R}^n \times h^{1+\beta}(\mathbb{T}^n) \rightarrow \mathcal{L}(\mathbb{R}^{n+1} \times h^{1+\gamma}(\mathbb{T}^n), \mathbb{R}^{n+1} \times h^{\gamma-\alpha}(\mathbb{T}^n))$$

is the operator defined by

$$\Psi(U_1, U_2, U_3)[(V_1, V_2, V_3)] := (0, 0, U_1 \Phi(U_3)[V_3] + U_2 \cdot \nabla V_3). \quad (3.53)$$

Since  $\nabla$  is a first order operator and  $-\Phi(U_3) \in \mathcal{H}(h^{1+\gamma}(\mathbb{T}^n), h^{\gamma-\alpha}(\mathbb{T}^n))$ , we infer from (1.6) and [2, Corollary I.1.6.3] that

$$-\Psi \in C^\infty((0, \infty) \times \mathbb{R}^n \times h^{1+\beta}(\mathbb{T}^n), \mathcal{H}(\mathbb{R}^{n+1} \times h^{1+\gamma}(\mathbb{T}^n), \mathbb{R}^{n+1} \times h^{\gamma-\alpha}(\mathbb{T}^n))).$$

Hence, we may argue as above to conclude that the quasilinear parabolic problem (3.52) has for each  $U_0 = (\lambda, u_0) \in (0, \infty) \times \mathbb{R}^n \times h^{1+\beta}(\mathbb{T}^n)$  a unique maximal solution  $U = U(\cdot; U_0)$  with  $(U_1, U_2) = \lambda_2$  and

$$U_3 \in C([0, T^+(U_0)), h^{1+\beta}(\mathbb{T}^n)) \cap C((0, T^+(U_0)), h^{1+\gamma}(\mathbb{T}^n)) \cap C^1((0, T^+(U_0)), h^{\gamma-\alpha}(\mathbb{T}^n)).$$

In addition,

$$\Omega := \{(\lambda, u_0, t) : t \in (0, T^+(\lambda, u_0))\}$$

is an open subset of  $\mathbb{R}^{n+1} \times h^{1+\beta}(\mathbb{T}^n) \times (0, \infty)$  and

$$[(\lambda, u_0, t) \mapsto U(t; (\lambda, u_0))] : \Omega \rightarrow \mathbb{R}^{n+1} \times h^{1+\beta}(\mathbb{T}^n)$$

is a smooth mapping. Therefore, given  $u_0 \in h^{1+\beta}(\mathbb{T}^n)$ , we have

$$\frac{T^+(u_0)}{\lambda_1} = T^+(\lambda, u_0) \quad \text{for } \lambda \in (0, \infty) \times \mathbb{R}^n.$$

In order to prove that  $u = u(\cdot; u_0)$  is smooth on  $(0, T^+(u_0)) \times \mathbb{R}^n$ , we choose an arbitrary point  $(t_0, x_0) \in (0, T^+(u_0)) \times \mathbb{R}^n$ . Let  $\mathbb{B}_\varepsilon \subset \mathbb{R}^{n+1}$  be the ball centered at  $(1, \dots, 1)$  of radius  $\varepsilon > 0$ , where we choose  $\varepsilon$  sufficiently small to guarantee that for all  $\lambda \in \mathbb{B}_\varepsilon$  we have  $t_0 < T^+(\lambda, u_0)$ . In particular, we have  $\mathbb{B}_\varepsilon \times \{u_0\} \times \{t_0\} \subset \Omega$  and the restriction

$$[\lambda \mapsto u_\lambda(t_0)] : \mathbb{B}_\varepsilon \rightarrow h^{1+\beta}(\mathbb{T}^n)$$

is smooth too, hence also the mapping

$$[\lambda \mapsto u_\lambda(t_0, x_0) = u(\lambda_1 t_0, x_0 - t_0 + \lambda_2 t_0)] : \mathbb{B}_\varepsilon \rightarrow \mathbb{R} \quad (3.54)$$

is smooth. Letting  $\mathbb{B}_\delta(t_0, x_0)$  be the ball centered at  $(t_0, x_0)$  with sufficiently small radius  $\delta$ , we note that the function  $\varphi : \mathbb{B}_\delta(t_0, x_0) \rightarrow \mathbb{B}_\varepsilon$ , defined by

$$\varphi(t, x) := \left( \frac{t}{t_0}, \frac{x - x_0 + t_0}{t_0} \right), \quad (3.55)$$

is well-defined and smooth. Composing the functions defined in (3.54) and (3.55), we conclude that

$$[(t, x) \mapsto u(t, x)] : \mathbb{B}_\delta(t_0, x_0) \rightarrow \mathbb{R}$$

is indeed smooth. This yields Theorem 1.1 (ii).

The proof for the a priori estimates of Theorem 1.1 (iii) is provided in Lemma 4.3 below.  $\square$

#### 4. STABILITY ANALYSIS

This section is devoted to the proof of Theorem 1.3. Let us emphasize that since each constant is a stationary solution to (1.4), the linearization of the right-hand side of (1.4) at any constant solution has zero as an eigenvalue, and this impedes us to applying directly (some version of the) the principle linearized stability in the context of (1.4). In order to establish our stability result presented in Theorem 1.3, we introduce a volume preserving unknown solving a quasilinear evolution problem, see (4.4) and (4.5) below, to which the quasilinear principle of linearized stability established in [33, Theorem 1.3] applies. Doing this, we derive in (4.6) and (4.9) estimates for the volume preserving unknown which are key to prove the exponential decay estimate (1.8).

We begin this section by showing that the stationary solutions to (1.4) coincide with the constant functions, see Lemma 4.1 below. In Section 4.2 we provide the improved uniqueness statement still missing in the proof of Theorem 1.1 and some maximum principles for the nonlocal mean curvature flow. We conclude the section with the proof of the stability result stated in Theorem 1.3.

Throughout this section the Hölder exponents  $\alpha, \beta, \gamma$  are assumed to satisfy (1.5).

**4.1. Stationary Solutions to (1.4).** Recalling (1.1), it is straightforward to see that each constant is a stationary solution to (1.4). In order to establish that these are the only stationary solutions, we use an alternative expression for the nonlocal mean curvature  $H_\alpha(\Sigma)$  of the hypersurface  $\Sigma := \{(x, u(x)) : x \in \mathbb{R}^n\}$  provided, e.g., in [6, 27]. More precisely, for given  $u \in \mathbf{h}^{1+\gamma}(\mathbb{T}^n)$ , we define for the sake of brevity  $H_\alpha(u) : \mathbb{R}^n \rightarrow \mathbb{R}$  by the formula

$$H_\alpha(u)(x) := H_\alpha(\Sigma)(x, u(x)), \quad x \in \mathbb{R}^n,$$

and note that, using Gauss' formula, one may express  $H_\alpha(u)$  as

$$H_\alpha(u)(x) = -\text{PV} \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\alpha}} \left[ F\left(\frac{\delta_{[x,y]}u}{|y|}\right) - F\left(-\frac{\delta_{[x,y]}u}{|y|}\right) \right] dy, \quad x \in \mathbb{R}^n, \quad (4.1)$$

where the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$F(\xi) := \int_\xi^\infty \frac{d\tau}{(1+\tau^2)^{\frac{n+1+\alpha}{2}}}, \quad \xi \in \mathbb{R}.$$

The following result identifies the stationary solutions as the constant constants and relies on [27, Lemma 3.6].

**Lemma 4.1.** *Let  $u \in C^{1+\gamma}(\mathbb{T}^n)$  be such that  $H_\alpha(u) = 0$ . Then  $u$  is a constant.*

*Proof.* Let  $m := \min_{\mathbb{R}^n} u$  and assume that  $v = u - m$  is not identically zero. Then, there exists  $x_0 \in \mathbb{R}^n$  such that  $v(x_0) = \max_{\mathbb{R}^n} v > 0$ . In view of  $H_\alpha(u) = 0$  and (4.1) we get

$$\text{PV} \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\alpha}} \left[ F\left(\frac{\delta_{[x_0,y]}v}{|y|}\right) - F\left(-\frac{\delta_{[x_0,y]}v}{|y|}\right) \right] dy = 0. \quad (4.2)$$

Moreover, the mean value theorem and  $F' < 0$  imply for  $y \neq 0$  that

$$F\left(\frac{\delta_{[x_0,y]}v}{|y|}\right) - F\left(-\frac{\delta_{[x_0,y]}v}{|y|}\right) = 2 \frac{v(x_0) - v(x_0 - y)}{|y|} \int_0^1 F'\left(s \frac{\delta_{[x_0,y]}v}{|y|} - (1-s) \frac{\delta_{[x_0,y]}v}{|y|}\right) ds \leq 0,$$

the latter expression being (strictly) negative when  $x_0 - y$  is close to a point where  $v$  attains the minimum. This contradicts (4.2), hence  $v = 0$  and the claim follows.  $\square$

**4.2. Maximum Principles for (1.4).** Using the formulation (4.1) for  $H_\alpha(u)$ , we establish next the uniqueness of solutions to (1.4) and to (3.51) within the regularity class (3.49) as announced in the proof of Theorem 1.1.

**Lemma 4.2.** *Let  $\lambda_1 > 0$ ,  $\lambda_2 \in \mathbb{R}^n$ , and  $T > 0$ . If*

$$u_1, u_2 \in C([0, T], h^{1+\beta}(\mathbb{T}^n)) \cap C((0, T), h^{1+\gamma}(\mathbb{T}^n)) \cap C^1((0, T), h^{\gamma-\alpha}(\mathbb{T}^n))$$

are solutions to

$$\frac{du}{dt}(t) = \lambda_1 \Phi(u(t))[u(t)] + \lambda_2 \cdot \nabla u(t), \quad t \in (0, T), \quad u(0) = u_0, \quad (4.3)$$

then  $u_1 = u_2$ .

*Proof.* Set  $v := u_1 - u_2$  and assume for contradiction that  $v \neq 0$  in  $[0, T'] \times \mathbb{R}^n$  for some  $T' \in (0, T)$ . Therefore, given  $\varepsilon > 0$ , also the function

$$w(t, x) := e^{-t\varepsilon} v(t, x), \quad (t, x) \in [0, T'] \times \mathbb{R}^n,$$

is not constant. Observing that  $w \in C([0, T'] \times \mathbb{R}^n)$ , we may assume that  $w$  attains a positive maximum at  $(t_0, x_0) \in (0, T'] \times \mathbb{R}^n$ ; that is  $w(t, x) \leq w(t_0, x_0)$  for all  $(t, x) \in [0, T'] \times \mathbb{R}^n$ . Since  $w \in C^1((0, T) \times \mathbb{R}^n)$  we deduce that  $\partial_t w(t_0, x_0) \geq 0$  and  $\nabla w(t_0, x_0) = \nabla v(t_0, x_0) = 0$ . These relations and the fact that  $u_1$  and  $u_2$  are solutions to (4.3) lead to

$$0 \leq \partial_t w(t_0, x_0) = -\varepsilon w(t_0, x_0) + e^{-\varepsilon t_0} \partial_t v(t_0, x_0) < e^{-\varepsilon t_0} \partial_t v(t_0, x_0),$$

where

$$\partial_t v(t_0, x_0) = \lambda_1 (1 + |\nabla u_1|^2)^{1/2}(t_0, x_0) (H_\alpha(u_2(t_0))(x_0) - H_\alpha(u_1(t_0))(x_0)),$$

and, in view of (4.1),

$$\begin{aligned} & H_\alpha(u_2(t_0))(x_0) - H_\alpha(u_1(t_0))(x_0) \\ &= \text{PV} \int_{\mathbb{R}^n} \frac{\delta_{[x_0,y]}v(t_0)}{|y|^{n+1+\alpha}} \int_0^1 F'\left((1-s) \frac{\delta_{[x_0,y]}u_2(t_0)}{|y|} + s \frac{\delta_{[x_0,y]}u_1(t_0)}{|y|}\right) ds dy \\ &+ \text{PV} \int_{\mathbb{R}^n} \frac{\delta_{[x_0,y]}v(t_0)}{|y|^{n+1+\alpha}} \int_0^1 F'\left(- (1-s) \frac{\delta_{[x_0,y]}u_2(t_0)}{|y|} - s \frac{\delta_{[x_0,y]}u_1(t_0)}{|y|}\right) ds dy. \end{aligned}$$

Since  $F' < 0$  and  $v(t_0, x_0) \geq v(t_0, x_0 - y)$  for all  $y \in \mathbb{R}^n$ , the latter expression is non-positive, and we obtain a contradiction.  $\square$

In Lemma 4.3 we present several maximum principles for the evolution problem (1.4). A similar result can be found, e.g., in [6, Lemma 5.3], but since these features are very special properties of the nonlocal mean curvature flow we recall them here.

**Lemma 4.3.** *Let  $u_0 \in \mathbf{h}^{1+\beta}(\mathbb{T}^n)$ . Then, the maximal solution  $u : [0, T^+) \rightarrow \mathbf{h}^{1+\beta}(\mathbb{T}^n)$  to (1.4) found in Theorem 1.1 satisfies*

- (i)  $\|u(t)\|_0 \leq \|u_0\|_0$  for  $t \in [0, T^+)$ .
- (ii)  $\|\partial_{x_i} u(t)\|_0 \leq \|\partial_{x_i} u_0\|_0$  for  $t \in [0, T^+)$  and  $1 \leq i \leq n$ .
- (iii) If  $\beta > \alpha$ , then  $\|\partial_t u(t)\|_0 \leq \|\partial_t u(0)\|_0$  for  $t \in [0, T^+)$ .

*Proof.* (i) We show that  $\|u(t)\|_0 < \|u_0\|_0 + \varepsilon$  for all  $t \in [0, T^+)$  and  $\varepsilon \in (0, 1)$ . To prove this claim we argue by contradiction and assume there was  $\varepsilon \in (0, 1)$  such that

$$\inf\{t \in [0, T^+) : \|u(t)\|_0 \geq \|u_0\|_0 + \varepsilon\} =: t_0 \in (0, T^+).$$

Then  $\|u(t_0)\|_0 = \|u_0\|_0 + \varepsilon$ , hence there exists  $x_0 \in \mathbb{R}^n$  such that  $|u(t_0, x_0)| = \|u_0\|_0 + \varepsilon$ . Without loss of generality we may assume that  $u(t_0, x_0) > 0$ , hence  $u$  attains the global maximum in  $[0, t_0] \times \mathbb{R}^n$  at  $(t_0, x_0)$ . Taking  $\partial_t u(t_0, x_0) \geq 0$  and  $\nabla u(t_0, x_0) = 0$  into account we get

$$0 \leq \partial_t u(t_0, x_0) = \text{PV} \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\alpha}} \left[ F\left(\frac{\delta_{[x_0, y]} u(t_0)}{|y|}\right) - F\left(-\frac{\delta_{[x_0, y]} u(t_0)}{|y|}\right) \right] dy < 0,$$

provided that  $u(t_0)$  is not constant. The strict inequality in the latter relation follows (when  $u(t_0)$  is not constant) by arguing as Lemma 4.1. Hence, if  $u(t_0)$  is not constant we obtain a contradiction.

If  $u(t_0)$  is constant, then there exists

$$t_1 := \inf\{t \in [0, T^+) : \|u(t)\|_0 \geq \|u_0\|_0 + \varepsilon/2\} \in (0, t_0).$$

On the one hand, if  $u(t_1)$  is not constant, we may argue as above to obtain a contradiction. On the other hand, if  $u(t_1)$  is constant, then  $u(t) = u(t_1)$  for all  $t \in (t_1, T^+)$  as constants are stationary solutions, hence also  $u(t_0) = u(t_1)$ , which is again a contradiction. This proves (i).

(ii) Differentiating (1.4) (with  $H_\alpha(u)$  as expressed in (4.1)), it follows for  $1 \leq j \leq n$  that the partial derivative  $u_j = \partial_{x_j} u$  satisfies

$$\begin{aligned} \partial_t u_j(t, x) &= -\partial_{x_j} (1 + |\nabla u|^2)^{1/2}(t, x) H_\alpha(u(t))(x) \\ &\quad + ((1 + |\nabla u|^2)^{1/2})(t, x) \\ &\quad \times \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\alpha}} \left[ \frac{\delta_{[x, y]} u_j(t)}{|y|} F'\left(\frac{\delta_{[x, y]} u(t)}{|y|}\right) + \frac{\delta_{[x, -y]} u_j(t)}{|y|} F'\left(-\frac{\delta_{[x, -y]} u(t)}{|y|}\right) \right] dy. \end{aligned}$$

Proceeding analogously as in (i) and noticing that

$$\partial_{x_j} (1 + |\nabla u|^2)^{1/2}(t_0, x_0) = 0$$

at a point  $(t_0, x_0) \in (0, T^+) \times \mathbb{R}^n$  with  $|u_j(t_0, x_0)| = \|u_j(t_0)\|_0$ , we deduce (ii).

(iii) This follows as above by differentiating (1.4) with respect to  $t$ .  $\square$

**4.3. Proof of Theorem 1.3.** We first introduce some notation. Given an integrable function  $u : \mathbb{T}^n \rightarrow \mathbb{R}$ , we denote by  $\langle u \rangle$  the integral mean of  $u$ ; that is,

$$\langle u \rangle := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} u \, dx.$$

Furthermore, we let  $P$  be the projection

$$Pu := \langle u \rangle.$$

Then, given  $r \geq 0$ , we can represent  $\mathfrak{h}^r(\mathbb{T}^n)$  as a direct sum

$$\mathfrak{h}^r(\mathbb{T}^n) = P\mathfrak{h}^r(\mathbb{T}^n) \oplus (I - P)\mathfrak{h}^r(\mathbb{T}^n),$$

where  $(I - P)\mathfrak{h}^r(\mathbb{T}^n)$  is the subspace of  $\mathfrak{h}^r(\mathbb{T}^n)$  consisting of those functions with zero integral mean.

*Proof of Theorem 1.3.* Let  $u : [0, T^+) \rightarrow \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$  be the maximal solution to

$$\frac{du}{dt}(t) = \Phi(u(t))[u(t)], \quad t > 0,$$

determined by  $u(0) = u_0 \in \mathfrak{h}^{1+\beta}(\mathbb{T}^n)$ , as found in Theorem 1.1. Given  $t \in [0, T^+)$ , let

$$q(t) := \langle u(t) \rangle \quad \text{and} \quad v(t) := u(t) - q(t). \quad (4.4)$$

Then  $v(0) = u_0 - \langle u_0 \rangle \in (I - P)\mathfrak{h}^{1+\beta}(\mathbb{T}^n)$  and  $v(t)$  and  $u(t)$  differ just by a  $t$ -dependent constant if  $t > 0$ . Recalling the definition of  $\Phi$ , we therefore have  $\Phi(u(t))[u(t)] = \Phi(v(t))[v(t)]$  for  $t \in [0, T^+)$ . The latter observation, the properties of a solution to (1.4), and (4.4) lead us now to

$$\begin{aligned} \frac{dv}{dt}(t) &= \frac{du}{dt}(t) - \left\langle \frac{du}{dt}(t) \right\rangle = \Phi(u(t))[u(t)] - \langle \Phi(u(t))[u(t)] \rangle \\ &= \Phi(v(t))[v(t)] - \langle \Phi(v(t))[v(t)] \rangle, \quad t > 0. \end{aligned}$$

We now define the operator

$$\Psi(u)[v] := \Phi(u)[v] - \langle \Phi(u)[v] \rangle,$$

and infer from (2.2) that

$$\Psi \in C^\infty((I - P)\mathfrak{h}^{1+\beta}(\mathbb{T}^n), \mathcal{L}(E_1, E_0)),$$

where

$$E_0 := (I - P)\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n) \quad \text{and} \quad E_1 := (I - P)\mathfrak{h}^{1+\gamma}(\mathbb{T}^n).$$

With this notation, we have identified  $v$  as a solution to the evolution problem

$$\frac{dv}{dt}(t) = \Psi(v(t))[v(t)], \quad t > 0, \quad v(0) = v_0 := u_0 - \langle u_0 \rangle, \quad (4.5)$$

which is quasilinear and of parabolic type. Indeed, with respect to the decompositions

$$\mathfrak{h}^{1+\gamma}(\mathbb{T}^n) = P\mathfrak{h}^{1+\gamma}(\mathbb{T}^n) \oplus E_1 \quad \text{and} \quad \mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n) = P\mathfrak{h}^{\gamma-\alpha}(\mathbb{T}^n) \oplus E_0,$$

we can represent  $\Phi(u)$  as a matrix operator

$$\Phi(u) = \begin{pmatrix} 0 & \langle \Phi(u)[\cdot] \rangle \\ 0 & \Psi(u) \end{pmatrix}.$$

This representation together with Theorem 3.1 and [2, Corollary I.1.6.3] implies that

$$-\Psi \in C^\infty((I - P)h^{1+\beta}(\mathbb{T}^n), \mathcal{H}(E_1, E_0)).$$

Moreover,  $v_* := 0$  is a stationary solution to (4.5) and the linearized operator

$$\mathbb{A} := \Psi(v_*) + (\partial\Psi(v_*)[\cdot])[v_*] = \Psi(v_*) \in \mathcal{L}(E_1, E_0)$$

is, in view of (3.3), (3.22), (3.23), and (3.26) the Fourier multiplier with symbol

$$m(k) = -\omega_0|k|^{1+\alpha}, \quad k \in \mathbb{Z}^n \setminus \{0\},$$

where the negative constant  $-\omega_0$  is defined by the integral (3.26) with  $a = 0$  (which is thus constant in  $x$ ). Since the embedding  $E_1 \hookrightarrow E_0$  is compact, we infer from [30, Theorem III.6.29] that the spectrum  $\sigma(\mathbb{A})$  consists entirely of isolated eigenvalues with finite algebraic multiplicities. Consequently,

$$\sigma(\mathbb{A}) = \{-\omega_0|k|^{1+\alpha} : k \in \mathbb{Z}^n \setminus \{0\}\}.$$

Hence,  $\operatorname{Re} \lambda \leq -\omega_0 < 0$  for all  $\lambda \in \sigma(\mathbb{A})$  and we are in a position to apply the quasilinear principle of linearized stability [33, Theorem 1.3] in the context of (4.5). Therefore, given  $\omega \in (0, \omega_0)$ , there exist constants  $\varepsilon > 0$  and  $M \geq 1$  such that for each  $\|v_0\|_{1+\beta} \leq \varepsilon$ , the solution  $v = v(\cdot; v_0)$  to (4.5) exists globally and

$$\|v(t)\|_{1+\beta} \leq M e^{-\omega t} \|v_0\|_{1+\beta}, \quad t \geq 0. \quad (4.6)$$

As already mentioned in the discussion subsequent to Theorem 1.1, we point out again that the nonlocal mean curvature of order  $\alpha$  is not well-defined for  $\beta \leq \alpha$ . The property (4.6) can thus not be used directly to derive estimates for  $\Phi(v(t))[v(t)]$ ,  $t > 0$ . Therefore, in order to establish the convergence of  $q(t)$  towards a constant we revisit the proof of [33, Theorem 1.3] to estimate  $\|v(t)\|_{1+\gamma}$  for  $t > 0$ .

To start with, we set

$$A(t) := -\Psi(v(t)), \quad t \geq 0.$$

Similarly as in the proof of Theorem 1.1, we choose a Hölder exponent  $\beta' \in (0, \beta)$  such that  $\gamma < \alpha + \beta'$ . We further set  $\rho := (1 + \alpha)^{-1}(\beta - \beta')$  and  $4\delta := \omega_0 - \omega$ . As shown in the proof of [33, Theorem 1.3], if  $\|v_0\|_{1+\beta} \leq \varepsilon$ , the solution  $v = v(\cdot; v_0)$  to (4.5) additionally satisfies

$$\|A(t) - A(s)\|_{\mathcal{L}(E_1, E_0)} \leq c \|v_0\|_{1+\beta} |t - s|^\rho, \quad t, s \in [0, \infty), \quad (4.7)$$

and

$$-\omega_0 + \delta + A(t) \in \mathcal{H}(E_1, E_0, \kappa, \delta), \quad t \in [0, \infty), \quad (4.8)$$

with a fixed constant  $\kappa \geq 1$ , see Section 2.1.

In view of (4.7)-(4.8) and the interpolation relation (2.1) we may apply [2, Lemma II.5.1.3] to deduce that the evolution operator  $U_A$  associated with the mapping

$$[t \mapsto A(t)] \in C^\rho([0, \infty), \mathcal{L}(E_1, E_0))$$

satisfies, after choosing a smaller  $\varepsilon$  if necessary, the estimate

$$\|U_A(t, s)\|_{\mathcal{L}((I-P)h^{1+\beta}(\mathbb{T}^n), E_1)} \leq c e^{-\omega(t-s)} (t-s)^{-\rho'}, \quad 0 \leq s < t < \infty,$$

where  $\rho' := (1 + \alpha)^{-1}(\gamma - \beta)$ . From this and [2, Remark II.2.1.2] we deduce that

$$\|v(t)\|_{1+\gamma} = \|U_A(t, 0)v_0\|_{1+\gamma} \leq c e^{-\omega t} t^{-\rho'} \|v_0\|_{1+\beta}, \quad t > 0. \quad (4.9)$$

We now consider the function  $q$  representing the integral mean of  $u$  for which we have

$$q'(t) = \left\langle \frac{du}{dt}(t) \right\rangle = \langle \Phi(u(t))[u(t)] \rangle = \langle \Phi(v(t))[v(t)] \rangle, \quad t > 0.$$

This implies that  $q(t)$  exists for all  $t \geq 0$ . Since  $q$  is continuous on  $[0, \infty)$  we also have

$$\langle u_0 \rangle = q(0) = \lim_{t \rightarrow 0} q(t)$$

and hence

$$\langle u(t) \rangle = q(t) = \langle u_0 \rangle + \int_0^t \langle \Phi(v(s))[v(s)] \rangle ds, \quad t \geq 0. \quad (4.10)$$

Indeed, the definition of  $\Phi$ , (2.6), and (4.9) imply for  $x \in \mathbb{R}^n$  and  $t > 0$  that

$$|\Phi(v(t))[v(t)](x)| \leq c \int_{\mathbb{R}^n} \frac{|\delta_{[x,y]}v(t) - y \cdot \nabla v(t, x - y)|}{|y|^{n+1+\alpha}} dy \leq ce^{-\omega t} t^{-\rho'} \|v_0\|_{1+\beta},$$

and therefore

$$\int_0^\infty |\langle \Phi(v(s))[v(s)] \rangle| ds \leq c \|v_0\|_{1+\beta} \int_0^\infty e^{-\omega s} s^{-\rho'} ds \leq c \|v_0\|_{1+\beta}. \quad (4.11)$$

This ensures in particular the convergence of the integral in (4.10). Moreover, given  $t \geq 1$ , we may estimate

$$\int_t^\infty |\langle \Phi(v(s))[v(s)] \rangle| ds \leq c \|v_0\|_{1+\beta} \int_t^\infty e^{-\omega s} ds \leq ce^{-\omega t} \|v_0\|_{1+\beta}. \quad (4.12)$$

Introducing the constant

$$C(u_0) := \lim_{t \rightarrow \infty} q(t) = \langle u_0 \rangle + \int_0^\infty \langle \Phi(v(s))[v(s)] \rangle ds \quad (4.13)$$

we infer from (4.10)-(4.12) that

$$|\langle u(t) \rangle - C(u_0)| \leq \int_t^\infty |\langle \Phi(v(s))[v(s)] \rangle| ds \leq Me^{-\omega t} \|u_0 - \langle u_0 \rangle\|_{1+\beta}, \quad t \geq 0, \quad (4.14)$$

for a possibly larger constant  $M \geq 1$ . Gathering (4.4), (4.6), and (4.14) implies (1.8). Finally, Lemma 4.3 (i) ensures that  $|C(u_0)| \leq \|u_0\|_0$ . This yields Theorem 1.3.  $\square$

#### APPENDIX A. A LIFTING PROPERTY FOR PERIODIC BESOV SPACES ON $\mathbb{T}^n$

We recall the definition of Besov spaces and refer to [37] for more details. As usual,  $\mathcal{D}'(\mathbb{T}^n)$  is the topological dual of  $\mathcal{D}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$ , and any  $f \in \mathcal{D}'(\mathbb{T}^n)$  can be represented as

$$f = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) e^{ik \cdot x},$$

where

$$\widehat{f}(k) := (2\pi)^{-n} f(e^{-ik \cdot x}), \quad k \in \mathbb{Z}^n.$$

Let  $(\varphi_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$  be such that

$$\text{supp } \varphi_0 \subset \{|x| \leq 2\}, \quad \text{supp } \varphi_j \subset \{2^{j-1} \leq |x| \leq 2^{j+1}\}$$

for  $j \geq 1$  with

$$\sum_{j=0}^{\infty} \varphi_j = 1 \quad \text{on } \mathbb{R}^n$$

and, for each  $\mu \in \mathbb{N}^n$ ,

$$\sup_{j \geq 0} \sup_{x \in \mathbb{R}^n} 2^{j|\mu|} |\partial^\mu \varphi_j(x)| < \infty.$$

Given  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , the Besov space  $B_{p,q}^s(\mathbb{T}^n)$  consists of all  $f \in \mathcal{D}'(\mathbb{T}^n)$  for which the norm

$$\|f\|_{B_{p,q}^s(\mathbb{T}^n)} := \left\| \left( 2^{sj} \left\| \sum_{k \in \mathbb{Z}^n} \varphi_j(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n, dx)} \right)_{j \in \mathbb{N}} \right\|_{\ell_q} \quad (\text{A.1})$$

is finite. Then  $B_{p,q}^s(\mathbb{T}^n)$  equipped with the norm  $\|\cdot\|_{B_{p,q}^s(\mathbb{T}^n)}$  is a Banach space (with equivalent norms when replacing  $(\varphi_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$  by another family with the same properties), see [37, Section 3.5.1/Theorem 1]. In view of [37, Section 3.5.4] we may identify  $C^s(\mathbb{T}^n)$  with the Besov space  $B_{\infty,\infty}^s(\mathbb{T}^n)$  provided that  $s > 0$  with  $s \notin \mathbb{N}$ .

As a preparatory result we provide in Lemma A.1 a direct proof for the boundedness of the Fourier multiplier  $I_t$  defined by the symbol  $\mathbf{m}_t(k) = |k|^t$  for  $k \neq 0$  and  $\mathbf{m}_t(0) = 1$  from  $B_{p,q}^s(\mathbb{T}^n)$  to  $B_{p,q}^{s-t}(\mathbb{T}^n)$  when  $s, t \in \mathbb{R}$ . In particular, Lemma A.1 shows that  $I_t$  is an isomorphism with inverse  $I_{-t}$ . A similar lifting property is established in the nonperiodic case in [38, Theorem 2.3.8]. As we are lacking a precise reference for the periodic case, we include a proof here.

**Lemma A.1.** *If  $s, t \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , then*

$$I_t := \left[ \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) e^{ik \cdot x} \mapsto \sum_{k \in \mathbb{Z}^n} \mathbf{m}_t(k) \widehat{f}(k) e^{ik \cdot x} \right] \in \mathcal{L}(B_{p,q}^s(\mathbb{T}^n), B_{p,q}^{s-t}(\mathbb{T}^n)),$$

where  $\mathbf{m}_t(k) = |k|^t$  for  $k \neq 0$  and  $\mathbf{m}_t(0) = 1$ .

The proof of Lemma A.1 is very much inspired by a result established for operator valued Fourier multipliers in the case  $n = 1$ , see [5]. Following [5, Proposition 2.2], we prove the following auxiliary result which is the basis for the proof of Lemma A.1.

**Lemma A.2.** *If  $\zeta \in \mathcal{S}(\mathbb{R}^n)$  and  $1 \leq p \leq \infty$ , then*

$$\left\| \sum_{|k| \leq M} \zeta(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)} \leq (2\pi)^{-n/2} \|\mathcal{F}^{-1} \zeta\|_{L_1(\mathbb{R}^n)} \|f\|_{L_p(\mathbb{T}^n)}$$

for every trigonometric polynomial  $f = \sum_{|k| \leq M} \widehat{f}(k) e^{ik \cdot x}$  with  $M \in \mathbb{N}$ .

*Proof.* Since the (nonperiodic) Fourier transform  $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$  is a bijection, we have

$$\begin{aligned} \left\| \sum_{|k| \leq M} \zeta(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)} &= (2\pi)^{-n/2} \left\| \int_{\mathbb{R}^n} \mathcal{F}^{-1} \zeta(\xi) \sum_{|k| \leq M} \widehat{f}(k) e^{ik \cdot (x-\xi)} d\xi \right\|_{L_p(\mathbb{T}^n)} \\ &= (2\pi)^{-n/2} \left\| \int_{\mathbb{R}^n} \mathcal{F}^{-1} \zeta(\xi) f(x-\xi) d\xi \right\|_{L_p(\mathbb{T}^n)}, \end{aligned}$$

and the claim follows from Young's inequality.  $\square$

We are now in a position to establish Lemma A.1.

*Proof of Lemma A.1.* We may assume that  $\text{supp } \varphi_0 \subset \{|x| \leq \sqrt{2}\}$ . Noticing that  $\mathbf{m}_t(k) = 1$  if  $|k| \leq \sqrt{2}$ , we have for  $j = 0$

$$2^{(s-t)j} \left\| \sum_{k \in \mathbb{Z}^n} \varphi_0(k) \mathbf{m}_t(|k|) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)} = 2^{sj} \left\| \sum_{|k| \leq 1} \varphi_0(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)}.$$

We then estimate the terms with  $j \geq 1$ . To this end we choose a function  $\psi \in \mathcal{S}(\mathbb{R})$  such that  $\text{supp } \psi \subset \{1/4 \leq |x| \leq 4\}$  and  $\psi = 1$  on  $\{1/2 \leq |x| \leq 2\}$ . Noticing that the function  $\zeta : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\zeta(x) := |x|^t \psi(|x|)$  belongs to  $\mathcal{S}(\mathbb{R}^n)$ , Lemma A.2 yields

$$\begin{aligned} & 2^{(s-t)j} \left\| \sum_{k \in \mathbb{Z}^n} \varphi_j(k) \mathbf{m}_t(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)} \\ &= 2^{sj} \left\| \sum_{2^{j-1} \leq |k| \leq 2^{j+1}} \psi(2^{-j}|k|) (2^{-j}|k|)^t \varphi_j(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)} \\ &\leq (2\pi)^{-n/2} \left\| \mathcal{F}^{-1}[x \mapsto \zeta(2^{-j}x)] \right\|_{L_1(\mathbb{R}^n)} 2^{sj} \left\| \sum_{k \in \mathbb{Z}^n} \varphi_j(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)} \\ &\leq (2\pi)^{-n/2} \left\| \mathcal{F}^{-1} \zeta \right\|_{L_1(\mathbb{R}^n)} 2^{sj} \left\| \sum_{k \in \mathbb{Z}^n} \varphi_j(k) \widehat{f}(k) e^{ik \cdot x} \right\|_{L_p(\mathbb{T}^n)}, \end{aligned}$$

where in the last step we used the property that

$$\left\| \mathcal{F}^{-1} \zeta \right\|_{L_1(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}[x \mapsto \zeta(bx)] \right\|_{L_1(\mathbb{R}^n)}$$

for  $b > 0$ . Recalling (A.1), the assertion follows.  $\square$

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