Occurrence of gap for one-dimensional scalar autonomous functionals with one end point condition

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Abstract

Let $L : \mathbb{R} \times \mathbb{R} \to [0, +\infty[\cup \{+\infty\}$ be a Borel function. We consider the problem

$$\min F(y) = \int_0^1 L(y(t), y'(t))\, dt : y(0) = 0,\ y \in W^{1,1}([0, 1]). \tag{P}$$

We give an example of a real valued Lagrangian L for which the Lavrentiev phenomenon occurs. We state a condition, involving only the behavior of L on the graph of two functions, that ensures the non-occurrence of the phenomenon. Our criterium weakens substantially the well-known condition, that L is bounded on bounded sets.

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1 Introduction

Consider the basic problem of the Calculus of Variations that consists on minimizing the autonomous integral functional

$$F(y) = \int_0^1 L(t, y(t), y'(t))\, dt$$

among the absolutely continuous functions on $[0, 1]$ that possibly satisfy some end-point conditions. Here $L : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[\cup\{+\infty\}$ is a Borel function. We are concerned with the question of avoiding the Lavrentiev phenomenon, namely the unpleasant fact that the infimum of F among the absolutely continuous functions is strictly less than the one among the Lipschitz functions that share the same end-point conditions. The occurrence of this phenomenon implies the failure of classical numerical analysis methods, e.g., finite elements, if one wishes to compute the infimum of F, and represents a discontinuity of F with respect to strong convergence in $W^{1,1}(I, \mathbb{R}^n)$.

Lavrentiev's phenomenon is considered among experts a matter of non-autonomous Lagrangians, i.e., depending explicitly on the time variable. On one side, a famous example by Manià exhibits the phenomenon when

$$L(t, y, v) \;=\; (y^3 - t)^2 v^6$$

among the functions $y : [0, 1] \to \mathbb{R}$ that satisfy the end-point conditions $y(0) = 0, y(1) = 1$ (see [4, §4.3]). A more refined construction by Ball and Mizel [2] shows that it may even occur when the Lagrangian is a polynomial in (t, y, v) that satisfies Tonelli's existence conditions (namely superlinearity and convexity in the last variable). On the other side, a celebrated result by Alberti and Serra Cassano [1, Theorem 2.4] asserts that non pathological autonomous Lagrangians do not exhibit the phenomenon. More precisely, there is no Lavrentiev phenomenon if

$$\forall K > 0 \quad \exists r_K > 0 \quad L(y, v) \text{ is bounded on } [-K, K]^n \times [-r_K, r_K]^n\,. \tag{B}$$

Notice that Condition (B) forces L to be finite on the union $\bigcup_{K>0}[-K, K]^n \times [-r_K, r_K]^n$ and in particular on $\mathbb{R}^n \times \{0\}$.

Actually, it has now become clear that the phenomenon is also strictly related to the presence and the number of end-point constraints. For instance, as shown in [4], Manià's example does not exhibit any more the phenomenon if one considers just the end-point condition $y(1) = 1$. Moreover, it was pointed out in [6] that

Condition (B) in [1, Theorem 2.4] is a sufficient condition for the non-occurrence of the phenomenon when one considers just one end-point condition, but not anymore in the case of two end-point conditions. In fact, Alberti provided an example (see [6, Example 3.5]) showing an autonomous Lagrangian with values either 0 or $+\infty$, satisfying (B) such that the functional F takes the value $+\infty$ on every Lipschitz function satisfying $y(0) = 0, y(1) = 1$.

Regarding the Lavrentiev phenomenon, the difference between one and two end-point conditions was recently better understood (see [6]). It seems to us of interest to study more thoroughly the conditions that provide the non-occurrence of the phenomenon for problems with one end-point condition. As mentioned above, Condition (B) of [1, Theorem 2.4] does not take into account the geometry of the effective domain $\mathrm{Dom}(L)$ of the Lagrangian (i.e., the set where it is finite). We wonder how sharp condition (B) is. Can it be weakened to an assumption involving just the subsets of the effective domain of L? The effort of finding such a condition was carried out in [5] under an additional convexity hypothesis along the radii from the origin on the last variable of the Lagrangian: it is enough in that case that for each $K > 0$, there is $r_K > 0$ such that $L(y, v)$ is bounded on $([-K, K]^n \times [-r_K, r_K]^n) \cap \mathrm{Dom}(L)$.

In this paper, we consider the case where the Lagrangian $L = L(y, v)$ is autonomous, $n = 1$, with the initial condition $y(0) = 0$ and free end-point condition. We first exhibit a finite, autonomous Lagrangian that violates (B), for which the Lavrentiev phenomenon occurs with just one end-point condition. We introduce the following Condition (R), weaker than (B), that ensures, with no need of any other additional hypothesis, the non-occurrence of the phenomenon:

Condition (R). There exist two locally Lipschitz functions ρ^-, ρ^+ defined on $\mathbb{R}$ such that:

$$\forall z \in \mathbb{R} \qquad \rho^-(z) < 0\,, \quad \rho^+(z) > 0\,,$$

and for every bounded interval J of $\mathbb{R}$,

$$\sup_{z\in J} L\big(z, \rho^-(z)\big) \;<\; +\infty\,, \quad \sup_{z\in J} L\big(z, \rho^+(z)\big) \;<\; +\infty\,.$$

The Condition (R) is fulfilled when (B) of [1, Theorem 2.4] holds. Condition (R) has the advantage to require the boundedness of the Lagrangian just on some one-dimensional subsets of its effective domain, without imposing, as (B) does, that $\mathrm{Dom}(L)$ contains the union of two-dimensional rectangles.

2 The functional, the gap and the phenomenon

2.1 The functional

For $1 \le p \le +\infty$ we will denote by $W^{1,p}(I)$ the Sobolev space of absolutely continuous functions $y : I \to \mathbb{R}$ such that $y' \in L^p(I)$; $\mathrm{Lip}(I) = W^{1,\infty}(I)$ is the space of Lipschitz functions on I.

We consider an autonomous Borel Lagrangian $L : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ with non–negative values, possibly infinite. We denote by I the unit interval $[0, 1]$ and we define

$$\forall y \in W^{1,1}(I) \qquad F(y) \;=\; \int_0^1 L(y(t), y'(t))\, dt.$$

We consider the end–point condition $y(0) = 0$ and the problem (P):

$$\min F(y) \;=\; \int_0^1 L(y(t), y'(t))\, dt\,, \quad y \in W^{1,1}(I)\,, \quad y(0) = 0\,. \tag{P}$$

2.2 The Lavrentiev gap and phenomenon

We focus our attention on the Lavrentiev gap and phenomenon problems with the prescribed boundary condition $y(0) = 0$. The following definition can be easily adapted to other kinds of boundary conditions.

Definition 2.1 (No gap)**.** Let $p \ge 1$. Let $y \in W^{1,p}(I)$ be such that $F(y) < +\infty$. We say that the *Lavrentiev gap* does not occur at y for (P) if there exists a sequence $(y_h)_{h\in\mathbb{N}}$ of functions satisfying:

1. for each $h \in \mathbb{N}$, the function y_h is Lipschitz and $y_h(0) = 0$;
2. $\lim\limits_{h\to+\infty} F(y_h) = F(y)$ (approximation in *energy*);
3. $y_h \to y$ in $W^{1,p}(I)$ (approximation in *norm*).

We denote by $\mathrm{Lip}(I)$ the space of the Lipschitz functions defined on I with values in $\mathbb{R}$.

Definition 2.2 (No phenomenon)**.** We say that there is no *Lavrentiev phenomenon* for (P) if

$$\inf(\mathrm{P}) \;=\; \inf\left\{ F(y) :\; y \in \mathrm{Lip}(I)\,, y(0) = 0 \right\}.$$

Remark 2.3. Clearly, the Lavrentiev phenomenon does not occur for (P) once there is no Lavrentiev gap for every $y \in W^{1,p}(I)$ such that $F(y) < +\infty$. Some others kinds of gap are of interest, namely between $W^{1,p}(I)$ and $W^{1,q}(I)$ for some $p < q < +\infty$. They are not considered here since our main result directly prevents the occurrence of the gap between $W^{1,1}(I)$ and $W^{1,+\infty}(I)$: the technique of the proof of Theorem 4.1 requires the functions ρ^+ and ρ^- to be locally Lipschitz.

3 Occurrence of the Lavrentiev gap for (P)

The Lavrentiev phenomenon is often considered a pathology related to non-autonomous Lagrangians. However, the phenomenon may also occur in the autonomous case: an example due to Alberti (see [6, Example 3.5]) exhibits an autonomous Lagrangian L that takes the value $+\infty$, satisfies (B), yet the Lavrentiev phenomenon occurs for a problem with two end-point conditions. When Condition (B) fails, the phenomenon may occur in the autonomous case, when one considers just one end-point condition. Consider

$$L(y,v) \;=\; \begin{cases} \left(v^2 - \dfrac{1}{4y^2}\right)^2 v^2 & \text{if } y \neq 0\,, \\ \qquad 1 & \text{if } y = 0\,. \end{cases}$$

The Lagrangian L is a Borel non–negative map. Let $y_*(s) = \sqrt{s}, s \in [0,1]$. Notice that (L, y_*) violates Condition (B) in [1]. Indeed if $v \neq 0$, then

$$\lim_{y \to 0} L(y,v) \;=\; +\infty\,,$$

and this implies that, for every $r > 0$, L is unbounded on $y_*(I) \times [-r, r]$.

Proposition 3.1. The function y_* is a minimizer for the problem (P) associated to L and moreover $F(y_*) = 0$. The gap occurs at y_*: more precisely, for any $y \in \mathrm{Lip}([0,1])$ such that $y(0) = 0$ we have $F(y) \geq 1$.

Proof. We have

$$\forall t > 0 \qquad y_*'(t) \;=\; \frac{1}{2\sqrt{t}} \;=\; \frac{1}{2y_*(t)}$$

so that $L(y_*(t), y_*'(t)) = 0$ for t in $]0,1]$. Moreover, we have $F(y) \geq 0$ for all admissible trajectory $y \in W^{1,1}([0,1])$, therefore

$$F(y_*) = 0 = \min\,(\mathrm{P})\,.$$

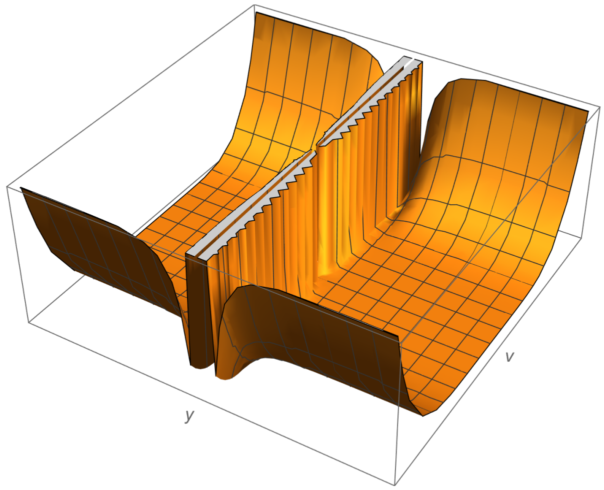

Figure 1: The graph of L in Proposition 3.1.

Next, let $y \in \mathrm{Lip}([0,1])$ with $y(0) = 0$ and assume that $F(y) < 1$. Notice that, since $F(0) = 1$, then y is not identically equal to 0. The set $\{t \in [0,1] : \ y(t) \neq 0\}$ being open and non-empty, it is a countable or finite union of non-empty open subintervals of $[0,1]$. Hence, there are $0 \leq a < b \leq 1$ such that

$$y(a) = 0, \quad y(b) \neq 0, \quad \forall t \in]a,b[\quad y(t) \neq 0\,.$$

Let $c \in]a,b[$. We have

$$\begin{aligned} F(y) &\geq \int_c^b L(y(t), y'(t))\,dt \\ &= \int_c^b \left(y'(t)^6 - \frac{1}{2}\frac{y'(t)^4}{y(t)^2} + \frac{1}{16}\frac{y'(t)^2}{y(t)^4} \right) dt \\ &\geq -\frac{1}{2}\int_c^b \frac{y'(t)^4}{y(t)^2}\,dt + \frac{1}{16}\int_c^b \frac{y'(t)^2}{y(t)^4}\,dt \\ &\geq -\frac{1}{2}\|y'\|_\infty^2 \int_c^b \frac{y'(t)^2}{y(t)^2}\,dt + \frac{1}{16}\int_c^b \frac{y'(t)^2}{y(t)^4}\,dt\,. \end{aligned} \tag{3.1}$$

Since $y(a) = 0$ and y is continuous at a, then $y(t) \to 0$ as $t \to a$, so that there exists $d \in]a,b[$ such that

$$\forall t \in]a,d] \qquad \frac{1}{2}\|y'\|_\infty^2 \frac{y'(t)^2}{y^2(t)} \leq \frac{1}{32}\frac{y'(t)^2}{y(t)^4}\,. \tag{3.2}$$

Now, in (3.1), fix $c \in]a, d[$. Integrating both terms of (3.2) over $[c, d]$, we obtain

$$-\frac{1}{2}\|y'\|_\infty^2 \int_c^d \frac{y'(t)^2}{y^2(t)}\,dt \;+\; \frac{1}{16}\int_c^d \frac{y'(t)^2}{y(t)^4}\,dt \;\geq\; \frac{1}{32}\int_c^d \frac{y'(t)^2}{y(t)^4}\,dt\,. \tag{3.3}$$

Inequalities (3.1) and (3.3) together yield

$$F(y) \;\geq\; -\frac{1}{2}\|y'\|_\infty^2 \int_d^b \frac{y'(t)^2}{y^2(t)}\,dt + \frac{1}{16}\int_d^b \frac{y'(t)^2}{y(t)^4}\,dt \;+\; \frac{1}{32}\int_c^d \frac{y'(t)^2}{y(t)^4}\,dt\,. \tag{3.4}$$

Notice that we use the lower bound (3.3) only for the integral over $[c, d]$. Jensen's inequality yields

$$\begin{aligned}\int_c^d \frac{y'(t)^2}{y(t)^4}\,dt \;\geq\; & \frac{1}{d-c}\left(\int_c^d \frac{y'(t)}{y(t)^2}\,dt\right)^2 \\ =\; & \frac{1}{d-c}\left(\frac{1}{y(c)} - \frac{1}{y(d)}\right)^2 .\end{aligned} \tag{3.5}$$

Since $y(a) = 0$ and y is continuous at a, we deduce from (3.5) that

$$\lim_{c\to a} \int_c^d \frac{y'(t)^2}{y(t)^4}\,dt \;=\; +\infty\,.$$

Keeping d fixed and taking the limit in (3.4) as c goes to a, we conclude that $F(y) = +\infty$, contradicting the initial assumption that $F(y) < 1$. □

Proposition 3.1 readily implies that the Lavrentiev phenomenon occurs for the problem with one end-point condition given by

$$\min \int_0^1 L(y(t), y'(t))\,dt\,, \quad y \in W^{1,1}(I)\,, \quad y(0) = 0\,.$$

The recent works [5, 6] have shown that the Lavrentiev phenomenon might be very sensitive to the number of prescribed end-point conditions. In fact, the same argument as above shows that the Lavrentiev phenomenon occurs for the problem with two end-point conditions given by

$$\min \int_0^1 L(y(t), y'(t))\,dt\,, \quad y \in W^{1,1}([0,1])\,, \quad y(0) = -1\,, y(1) = 1\,.$$

However, in the above example, if we keep only one of the two end-point conditions, the Lavrentiev phenomenon disappears! To see this, we proceed as in [4, §4.3].

4 Non-Occurrence of the Lavrentiev Gap and phenomenon for (P)

4.1 Non-occurrence of the gap

Let $L : \mathbb{R}\times\mathbb{R} \to [0, +\infty]$ be an autonomous Borel Lagrangian with non–negative values, possibly infinite. Let $p \geq 1$. For a given $y \in W^{1,p}(I)$, we consider the following condition.

Condition (R$_y$). There exist two Lipschitz functions ρ^-, ρ^+ defined on $y(I)$ such that

$$\forall z \in y(I) \qquad \rho^-(z) < 0\,, \quad \rho^+(z) > 0\,,$$
$$\sup_{z\in y(I)} L\big(z, \rho^-(z)\big) \,<\, +\infty\,, \quad \sup_{z\in y(I)} L\big(z, \rho^+(z)\big) \,<\, +\infty\,.$$

Theorem 4.1 (**Non-occurrence of the Lavrentiev gap**)**.** Let $y \in W^{1,p}([0, 1])$ be such that $F(y) < +\infty$. Assume that y satisfies Condition (R$_y$). Then there is no Lavrentiev gap for (P) at y.

Remark 4.2. Condition (R$_y$) weakens Assumption (B) formulated in [1]. Indeed if L is bounded on $y(I) \times [-r, r]$ for some $r > 0$, then (R$_y$) is satisfied with $\rho^- = -r$ and $\rho^+ = r$.

The strategy of the proof is the same as the proof of Alberti and Serra Cassano [1]. Yet it differs at some specific points and it requires also a different construction for the approximating function. In order to facilitate the reading, we have chosen to write the full proof. Another reason is that, as we work with real valued functions, some arguments become simpler than in the n–dimensional case. For convenience, we restate a general lemma of integration theory that were proved in [1].

Lemma 4.3. [1, Lemma 2.6] Let $g : I \to [0, +\infty]$ be a Lebesgue measurable function and let B_h be a sequence of measurable subsets of I such that $|I\backslash B_h| \to 0$ as $h \to \infty$. Then

$$\lim_{h\to+\infty} \int_{B_h} g\, dt \,=\, \int_I g\, dt\,.$$

We will need the following variation of [1, Lemma 2.7]. Compared with the original version we do not need here that the functions φ_h are Lipschitz nor that φ_h' are bounded from below on I, and the conclusion is weaker.

Lemma 4.4. Let $\varphi_h : I \to \mathbb{R}$, $h \in \mathbb{N}$, be a sequence of absolutely continuous functions with $\varphi_h' > 0$ a.e. on I and $(E_h)_{h\in\mathbb{N}}$ be a sequence of measurable sets of I such that:

1. For some $c > 0$, $\varphi_h' \geq c$ a.e. in E_h;
2. $\varphi_h(t) \to t$ as $h \to +\infty$ for every $t \in I$.

Then, for every $f \in L^p(\mathbb{R})$,

$$\lim_{h\to+\infty} \int_{\varphi_h^{-1}(I)\cap E_h} |f - f(\varphi_h)|^p \, d\tau = 0.$$

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of smooth functions such that $f_k \to f$ in $L^p(I)$. We write

$$\begin{aligned} &\|f - f(\varphi_h)\|_{L^p(\varphi_h^{-1}(I)\cap E_h)} \leq \\ &\qquad \|f - f_k\|_{L^p(I)} + \|f_k - f_k(\varphi_h)\|_{L^p(I)} + \|f_k(\varphi_h) - f(\varphi_h)\|_{L^p(\varphi_h^{-1}(I)\cap E_h)}. \end{aligned}$$

Of course, $\|f - f_k\|_{L^p(I)} \to 0$ in $L^p(I)$ and the dominated convergence theorem, for each k fixed, implies that $\|f_k - f_k(\varphi_h)\|_{L^p(I)} \to 0$ as $h \to +\infty$. It remains to study the convergence of $\|f_k(\varphi_h) - f(\varphi_h)\|_{L^p(\varphi_h^{-1}(I)\cap E_h)}$. For each $h \in \mathbb{N}$ let $\psi_h : \varphi_h(I) \to I$ be the inverse of φ_h; notice that by Banach-Zarecki's theorem ψ_h is absolutely continuous (see [3, 7]). The change of variables $t = \varphi_h(\tau)$ yields (see, for instance, [8]),

$$\begin{aligned} \int_{\varphi_h^{-1}(I)\cap E_h} |f_k(\varphi_h) - f(\varphi_h)|^p \, d\tau &\leq \frac{1}{c} \int_{\varphi_h^{-1}(I)\cap E_h} |f_k(\varphi_h) - f(\varphi_h)|^p \varphi_h' \, d\tau \\ &\leq \frac{1}{c} \int_{\varphi_h^{-1}(I)} |f_k(\varphi_h) - f(\varphi_h)|^p \varphi_h' \, d\tau \\ &= \frac{1}{c} \int_I |f_k(t) - f(t)|^p \, dt \\ &= \frac{1}{c} \|f_k(t) - f(t)\|^p_{L^p(I)} \, dt \to 0 \end{aligned}$$

as $k \to +\infty$. The conclusion follows. $\square$

We will use several times the fact that, since ρ^+, ρ^- are continuous functions on I, there are positive constants $\rho_{min} > 0, \rho_{max} > 0$ such that

$$\forall x \in I \qquad \min\{\rho^+(x), -\rho^-(x)\} \geq \rho_{min}, \quad \max\{\rho^+(x), -\rho^-(x)\} \leq \rho_{max}\,.$$

Proof of Theorem 4.1. We assume that y is not constant, otherwise the conclusion is trivial. We start by applying a classical result which is a consequence of Lusin's theorem (see for instance Theorem 3.10 in [9]). For every $h \in \mathbb{N}$, there are a Lipschitz function $u_h : I \to \mathbb{R}$ and an open subset A_h of I such that, denoting by $|A|$ the measure of a subset A of I:

1. $|A_h| \leq 1/h$,
2. $u_h = y$, $u_h' = y'$ in $I \setminus A_h$,
3. $u_h(0) = y(0)$, $u_h(1) = y(1)$,
4. u_h is affine in each connected component of A_h,
5. A_h is a countable union of disjoint open intervals $I_{h,k}$, $k \in J_h \subset \mathbb{N}$.

The set A_h is somehow the bad set where the function y might behave badly in the sense that its derivative might be unbounded on A_h.
We claim that it is not restrictive to assume that u_h' does not vanish on A_h. Indeed, assume the contrary. Each A_h is a union of disjoint open intervals $(I_{h,k})_{k \in J_h}$, and we may modify the sets A_h as follows:

- We first remove from A_h the intervals where y is itself constant on which, as a byproduct, $y = u_h$;
- On every other subinterval $I_{h,k} = (a_{h,k}, b_{h,k})$, $k \in J_h' \subset J_h$ where u_h is constant but y is not, we choose $c_{h,k} \in I_{h,k}$ such that $y(c_{h,k}) \neq y(a_{h,k})$. On

$$I_{h,k} \setminus \{c_{h,k}\} = (a_{h,k}, c_{h,k}) \cup (c_{h,k}, b_{h,k})$$

  we define $\tilde{u}_h$ to be affine in $(a_{h,k}, c_{h,k})$ joining $y(a_{h,k})$ to $y(c_{h,k})$ and affine in $(c_{h,k}, b_{h,k})$ joining $y(c_{h,k})$ to $y(b_{h,k})$. Namely

$$\forall \tau \in [a_{h,k}, c_{h,k}] \quad u_h(\tau) = y(a_{h,k}) + \frac{\tau - a_{h,k}}{c_{h,k} - a_{h,k}}(y(c_{h,k}) - y(a_{h,k})),$$

$$\forall \tau \in [c_{h,k}, b_{h,k}] \quad u_h(\tau) = y(c_{h,k}) + \frac{\tau - c_{h,k}}{b_{h,k} - c_{h,k}}(y(b_{h,k}) - y(c_{h,k})).$$

We then set

$$\tilde{A}_h = A_h \setminus \bigcup_{k \in J_h'} \{c_{h,k}\}.$$

Clearly $|A_h| = |\tilde{A}_h|$ and $(\tilde{u}_h, \tilde{A}_h)$ satisfy properties 1-5, proving the claim: we thus assume henceforth that u_h' does not vanish on A_h.
Notice that, since u_h is affine on every interval $I_{h,k}$ and u_h, y are equal at the extremities of $I_{h,k}$, then

$$\begin{aligned}\int_{A_h} |u_h'|\, d\tau &= \sum_{k\in J_h} \int_{I_{h,k}} |u_h'|\, d\tau = \sum_{k\in J_h} \left| \int_{I_{h,k}} u_h'\, d\tau \right| \\ &= \sum_{k\in J_h} \left| \int_{I_{h,k}} y'\, d\tau \right| \\ &\leq \sum_{k\in J_h} \int_{I_{h,k}} |y'|\, d\tau = \int_{A_h} |y'|\, d\tau \to 0 \quad \text{as} \quad h \to +\infty.\end{aligned} \tag{4.1}$$

The first problem with the function u_h is that we might have $F(u_h) = +\infty$, indeed, the integral of $L(u_h, u_h')$ over the intervals $I_{h,k}$ might very well be infinite. We shall take advantage of the functions ρ^-, ρ^+ to replace the portions of the function u_h over A_h by a function v_h having a finite energy on A_h, which in addition tends to 0 as $h \to +\infty$. We define a function ρ_h on A_h by setting

$$\forall \tau \in A_h \qquad \rho_h(\tau) = \begin{cases} \rho^+(u_h(\tau)) & \text{if } u_h'(\tau) > 0\,, \\ -\rho^-(u_h(\tau)) & \text{if } u_h'(\tau) < 0\,. \end{cases} \tag{4.2}$$

Notice that ρ_h is positive and continuous. In order to perform an adequate change of variable, we define next a function $\varphi_h \in W^{1,1}(I)$ by setting $\varphi_h(0) = 0$ and

$$\varphi_h'(\tau) = \begin{cases} 1 & \text{if } \tau \in I \setminus A_h\,, \\ \dfrac{|u_h'(\tau)|}{\rho_h(\tau)} & \text{if } \tau \in A_h\,. \end{cases} \tag{4.3}$$

Using inequality (4.1), we have

$$\begin{aligned}|\varphi_h(A_h)| &= \int_{\varphi_h(A_h)} 1\, dt = \int_{A_h} \varphi_h'(\tau)\, d\tau = \int_{A_h} \frac{|u_h'(\tau)|}{\rho_h(\tau)}\, d\tau \\ &\leq \frac{1}{\rho_{min}} \int_{A_h} |u_h'(\tau)|\, d\tau \to 0 \quad \text{as} \quad h \to +\infty\,.\end{aligned} \tag{4.4}$$

Next, we claim that the function φ_h converges uniformly towards the identity map on I. Indeed, $\varphi_h(0) = 0$ and moreover, from (4.4) and the fact that $|A_h| \to 0$, for all $t \in I$ we have

$$
\begin{aligned}
|\varphi_h(t) - t| &\leq \int_I |\varphi_h' - 1| \, d\tau \\
&\leq \int_{A_h} \left(\varphi_h'(\tau) + 1\right) \, d\tau \\
&= |\varphi_h(A_h)| + |A_h| \;\to\; 0 \qquad \text{as} \quad h \to +\infty \, .
\end{aligned}
$$

In particular, we have

$$
|\varphi_h(I)| \;=\; \int_I \varphi_h'(\tau) \, d\tau \;=\; \varphi_h(1) - \varphi_h(0) \;\to\; |I| = 1 \quad \text{as} \quad h \to +\infty \, .
$$

Setting $T_h = \varphi_h(1)$, we thus have $T_h \to 1$ as $h \to +\infty$. However we don't know whether T_h is smaller or larger than 1 and this will create some trouble later on. The derivative φ_h' is strictly positive on I, therefore φ_h is strictly increasing and it is a one to one map from $[0, 1]$ onto $[0, T_h]$. Its inverse $\psi_h : [0, T_h] \to [0, 1]$ is continuous, strictly increasing. Since $\varphi_h' > 0$ a.e. on I then ψ_h is absolutely continuous (see [7, Ch. IX]) and its derivative is given by

$$
\forall t \in [0, T_h] \qquad \psi_h'(t) \;=\; \frac{1}{\varphi_h'\big(\psi_h(t)\big)} \, .
$$

Using the expression of φ_h' given in (4.3), we obtain

$$
\psi_h'(t) \;=\; \begin{cases} 1 & \text{if } t \in [0, T_h] \setminus \varphi_h(A_h) \, , \\ \dfrac{\rho_h(\psi_h(t))}{|u_h'(\psi_h(t))|} & \text{if } t \in \varphi_h(A_h) \, . \end{cases} \tag{4.5}
$$

The change of variable $\tau = \psi_h(t)$ gives

$$
\begin{aligned}
1 = |I| \;\geq\; |\psi_h(I \cap [0, T_h])| \;&=\; \int_{\psi_h(I \cap [0, T_h])} 1 \, d\tau \;=\; \int_{I \cap [0, T_h]} \psi_h'(t) \, dt \\
&\geq\; \int_{(I \cap [0, T_h]) \setminus \varphi_h(A_h)} 1 \, dt \;\geq\; \min\{1, T_h\} - |\varphi_h(A_h)| \, ,
\end{aligned}
$$

so that

$$
\lim_{h \to +\infty} |\psi_h(I \cap [0, T_h])| \;=\; 1 \, . \tag{4.6}
$$

We define a new function v_h by setting

$$\forall t \in [0, T_h] \qquad v_h(t) = u_h(\psi_h(t)).$$

The function v_h, being the composition of the Lipschitz function u_h with the absolutely continuous function ψ_h, is absolutely continuous with derivative given by

$$\forall t \in [0, T_h] \qquad v_h'(t) \;=\; u_h'(\psi_h(t))\,\psi_h'(t)\,.$$

For $t \in [0, T_h] \setminus \varphi_h(A_h)$, we have $\psi_h'(t) = 1$ and $\psi_h(t) \notin A_h$, whence

$$u_h'(\psi_h(t))\,\psi_h'(t) \;=\; u_h'(\psi_h(t)) \;=\; y'(\psi_h(t)) \tag{4.7}$$

and thus

$$\forall t \in [0, T_h] \setminus \varphi_h(A_h) \qquad v_h'(t) \;=\; y'(\psi_h(t))\,.$$

Let next $t \in \varphi_h(A_h)$. In this case, we have

$$\begin{aligned} v_h'(t) \;&=\; \frac{u_h'(\psi_h(t))}{\varphi_h'(\psi_h(t))} \;=\; \frac{u_h'(\psi_h(t))}{|u_h'(\psi_h(t))|}\rho_h(\psi_h(t)) \\ &=\; \mathrm{sgn}(u_h'(\psi_h(t)))\,\rho_h(\psi_h(t))\,, \end{aligned} \tag{4.8}$$

where sgn is the classical sign function given by

$$\mathrm{sgn}(x) \;=\; \begin{cases} +1 & \text{if } x > 0\,, \\ -1 & \text{if } x < 0\,. \end{cases}$$

Recalling the definition (4.2) of ρ_h, we see that

$$\forall t \in \varphi_h(A_h) \qquad v_h'(t) \;=\; \begin{cases} \rho^+(u_h(\psi_h(t))) & \text{if} \quad u_h'(\psi_h(t)) > 0\,, \\ \rho^-(u_h(\psi_h(t))) & \text{if} \quad u_h'(\psi_h(t)) < 0\,. \end{cases}$$

This implies in particular that

$$\forall t \in \varphi_h(A_h) \qquad |v_h'(t)| \;\le\; \rho_{max}\,. \tag{4.9}$$

From formula (4.7), we see that v_h' is also bounded on $[0, T_h] \setminus \varphi_h(A_h)$, since the function u_h is Lipschitz. We conclude that v_h is Lipschitz on $[0, T_h]$.
Construction of the approximating sequence. We finally build the Lipschitz function w_h upon v_h on $[0, 1]$ which is the suitable approximation of y in energy and

in the space $W^{1,p}(I)$. Two cases may occur:
a) If $T_h \geq 1$, then we define w_h to be the restriction of v_h to $[0,1]$.
b) If $T_h < 1$, then we shall extend v_h from $[0,T_h]$ to $[0,1]$ as we explain next. We define

$$\alpha = \min y(I)\,, \qquad \beta = \max y(I)\,.$$

We first extend ρ^-, ρ^+ to Lipschitz functions $\widetilde{\rho}^-, \widetilde{\rho}^+$ on $\mathbb{R}$ defined by

$$\widetilde{\rho}^-(z) = \begin{cases} \rho^-(\alpha) & z < \alpha \\ \rho^-(z) & z \in [\alpha,\beta] \\ \rho^-(\beta) & z > \beta. \end{cases}, \quad \widetilde{\rho}^+(z) = \begin{cases} \rho^+(\alpha) & z < \alpha \\ \rho^+(z) & z \in [\alpha,\beta] \\ \rho^+(\beta) & z > \beta. \end{cases}$$

We consider the differential equation

$$z'(t) = \widetilde{\rho}^+(z(t))$$

and we denote by $z_1^+(t)$ the solution with initial condition $z(\tau_0) = y_0$, where $\tau_0 = T_h$ and $y_0 = v_h(T_h)$. We set

$$\tau_1 = \inf \left\{ t \geq \tau_0 : z_1^+(t) = \beta \right\}.$$

If $\tau_1 < 1$, then we set $w_h(t) = z_1^+(t)$ on $[\tau_0, \tau_1]$. We consider then the differential equation

$$z'(t) = \widetilde{\rho}^-(z(t))$$

and we denote by $z_2^-(t)$ the solution with initial condition $z(\tau_1) = z_1^+(\tau_1) = \beta$. We set

$$\tau_2 = \inf \left\{ t \geq \tau_1 : z_2^-(t) = \alpha \right\}.$$

Notice that, since $-\rho_{max} \leq \rho^-$, the travelling speed to go from β to α is at most ρ_{max} and thus

$$\tau_2 - \tau_1 \geq \frac{\beta - \alpha}{\rho_{max}}.$$

If $\tau_2 < 1$, then we extend $w_h(t)$ on $[\tau_1, \tau_2]$ by setting $w_h(t) = z_2^-(t)$ on this interval. We iterate this construction. Since at each stage $i \geq 1$, we have

$$\tau_{i+1} - \tau_i \geq \frac{\beta - \alpha}{\rho_{max}},$$

then the process ends at the first index m such that $\tau_m < 1 \leq \tau_{m+1}$, after a number m of steps that is bounded by a number depending only on $\beta - \alpha$ and $1 - T_h$. In fact, we have

$$m \leq \frac{\rho_{max}}{\beta - \alpha}(1 - T_h) + 1\,.$$

In the last step we extend $w_h(t)$ on $[\tau_m, 1]$ by restricting the solution of the differential equation to this interval. In what follows we set $\tau_{m+1} = 1$ for convenience. To sum up, we have

$$\forall t \in [0,1] \quad w_h(t) \in [\alpha, \beta], \quad w_h'(t) \;=\; \begin{cases} v_h'(t) & \text{if } t \in [0, T_h]\,, \\ \rho^+(w_h(t)) & \text{if } t \in [\tau_i, \tau_{i+1}]\,, \quad i \text{ even}\,, \\ \rho^-(w_h(t)) & \text{if } t \in [\tau_i, \tau_{i+1}]\,, \quad i \text{ odd}\,. \end{cases}$$

Notice that

$$\forall t \in [T_h, 1] \qquad |w_h'(t)| \;\leq\; \rho_{max}\,,$$

hence the function w_h is still Lipschitz.

We show next that w_h converges to y in $W^{1,p}(I)$. We decompose the integral as the sum of three terms

$$\|w_h' - y'\|^p_{L^p(I)} \;=\; \int_I |w_h'(t) - y'(t)|^p \, dt \;=\; P_{1,h} + P_{2,h} + P_{3,h}\,,$$

where, recalling that $w_h = v_h$ on $[0, T_h]$,

$$\begin{aligned} P_{1,h} \;&=\; \int_{(I \cap [0,T_h]) \setminus \varphi_h(A_h)} |v_h'(t) - y'(t)|^p \, dt\,, \\ P_{2,h} \;&=\; \int_{I \cap [0,T_h] \cap \varphi_h(A_h)} |v_h'(t) - y'(t)|^p \, dt\,, \\ P_{3,h} \;&=\; \int_{[\min\{T_h,1\},1]} |w_h'(t) - y'(t)|^p \, dt\,. \end{aligned}$$

We prove next that the three terms $P_{1,h}, P_{2,h}, P_{3,h}$ tend to 0 as $h \to +\infty$. We have

$$P_{1,h} \;=\; \int_{(I \cap [0,T_h]) \setminus \varphi_h(A_h)} |u_h'(\psi_h(t))\psi_h'(t) - y'(t)|^p \, dt\,.$$

It follows from (4.5) that $\psi_h' = 1$ on $(I \cap [0, T_h]) \setminus \varphi_h(A_h)$, therefore we can rewrite $P_{1,h}$ as

$$P_{1,h} \;=\; \int_{(I \cap [0,T_h]) \setminus \varphi_h(A_h)} |u_h'(\psi_h(t)) - y'(t)|^p \psi_h'(t) \, dt\,.$$

The change of variable $\tau = \psi_h(t)$ then yields

$$P_{1,h} \;=\; \int_{\psi_h(I \cap [0,T_h]) \setminus A_h} |u_h'(\tau) - y'(\varphi_h(\tau))|^p \, d\tau\,.$$

Using the fact that $u_h' = y'$ on $I \setminus A_h$, we obtain

$$P_{1,h} \;=\; \int_{\psi_h(I\cap[0,T_h])\setminus A_h} \big|y'(\tau) - y'(\varphi_h(\tau))\big|^p \, d\tau.$$

Notice that $\psi_h(I \cap [0,T_h]) \setminus A_h = \varphi_h^{-1}(I) \cap E_h$, with $E_h = I \setminus A_h$ and that $\varphi_h' = 1$ on E_h. By applying Lemma 4.4 we obtain that $P_{1,h} \to 0$ as $h \to +\infty$. Concerning $P_{2,h}$, we notice that

$$P_{2,h} \le 2^p \left(\int_{I\cap[0,T_h]\cap\varphi_h(A_h)} |v_h'(t)|^p \, dt + \int_{I\cap[0,T_h]\cap\varphi_h(A_h)} |y'(t)|^p \, dt \right).$$

It follows from (4.9) and (4.4) that

$$\int_{I\cap[0,T_h]\cap\varphi_h(A_h)} |v_h'(t)|^p \, dt \;\le\; (\rho_{max})^p \, |\varphi_h(A_h)| \to 0 \quad \text{as} \quad h \to +\infty,$$

and the integrability of $|y'|^p$ together with (4.4) immediately gives,

$$\int_{I\cap[0,T_h]\cap\varphi_h(A_h)} |y'(t)|^p \, dt \to 0 \quad \text{as} \quad h \to +\infty,$$

so that $P_{2,h} \to 0$ as $h \to +\infty$. Finally, we have

$$P_{3,h} \le 2^p \left(\int_{[\min\{T_h,1\},1]} |w_h'(t)|^p \, dt + \int_{[\min\{T_h,1\},1]} |y'(t)|^p \, dt \right).$$

As above, since $T_h \to 1$ as $h \to +\infty$, the integrability of $|y'|^p$ gives

$$\int_{[\min\{T_h,1\},1]} |y'(t)|^p \, dt \to 0 \quad \text{as} \quad h \to +\infty.$$

Moreover, $|w_h'| \le \rho_{max}$ on $I \cap [T_h, 1]$, so that

$$\int_{[\min\{T_h,1\},1]} |w_h'(t)|^p \, dt \;\le\; (\rho_{max})^p (1 - \min\{T_h,1\}) \to 0 \quad \text{as} \quad h \to +\infty.$$

We show now that $F(w_h)$ converges to $F(y)$ as $h \to +\infty$. By definition, we have

$$F(w_h) \;=\; \int_I L(w_h, w_h') \, dt\,.$$

We decompose the integral as the sum of three terms

$$F(w_h) \;=\; Q_{1,h} + Q_{2,h} + Q_{3,h}\,,$$

where, recalling that $w_h = v_h$ on $[0, T_h]$,

$$\begin{aligned}
Q_{1,h} \;&=\; \int_{(I\cap[0,T_h])\setminus\varphi_h(A_h)} L(v_h, v_h')\,dt\,,\\
Q_{2,h} \;&=\; \int_{I\cap[0,T_h]\cap\varphi_h(A_h)} L(v_h, v_h')\,dt\,,\\
Q_{3,h} \;&=\; \int_{[\min\{T_h,1\},1]} L(w_h, w_h')\,dt\,.
\end{aligned}$$

We prove next that $Q_{1,h}$ converges towards $F(y)$ while $Q_{2,h}, Q_{3,h}$ tend to 0 as $h \to +\infty$. From the definition of v_h, we have

$$Q_{1,h} \;=\; \int_{(I\cap[0,T_h])\setminus\varphi_h(A_h)} L\big(u_h(\psi_h(s)), u_h'(\psi_h(s))\psi_h'(s)\big)\,ds\,.$$

It follows from (4.5) that $\psi_h' \;=\; 1$ on $(I \cap [0, T_h]) \setminus \varphi_h(A_h)$, therefore we can rewrite $Q_{1,h}$ as

$$Q_{1,h} \;=\; \int_{(I\cap[0,T_h])\setminus\varphi_h(A_h)} L\big(u_h(\psi_h(s)), u_h'(\psi_h(s))\big)\,ds\,.$$

The change of variable $\tau = \psi_h(s)$ yields then

$$Q_{1,h} \;=\; \int_{\psi_h(I\cap[0,T_h])\setminus A_h} L\big(u_h(\tau), u_h'(\tau)\big)\,d\tau\,.$$

Using the fact that $u_h = y$ and $u_h' = y'$ on $I \setminus A_h$, we obtain

$$Q_{1,h} \;=\; \int_{\psi_h(I\cap[0,T_h])\setminus A_h} L\big(y(\tau), y'(\tau)\big)\,d\tau\,.$$

Lemma 4.3 and the estimate (4.6) allow to conclude that

$$Q_{1,h} \quad\to\quad \int_I L\big(y(\tau), y'(\tau)\big)\,d\tau \quad \text{as} \quad h \to +\infty\,.$$

Thus we are done with $Q_{1,h}$. We deal next with $Q_{2,h}$. The expression of the derivative of v_h' on $I \cap [0,T_h] \cap \varphi_h(A_h)$ was computed in (4.8), so we have

$$Q_{2,h} \;=\; \int_{I\cap[0,T_h]\cap\varphi_h(A_h)} L\Big(u_h(\psi_h(t)), \operatorname{sgn}(u_h'(\psi_h(t)))\,\rho_h(\psi_h(t))\Big)\,dt\,.$$

The change of variable $\tau = \psi_h(t)$ gives, with the help of the expression of ψ_h' computed in (4.5),

$$\begin{aligned} Q_{2,h} \;&=\; \int_{\psi_h(I\cap[0,T_h])\cap A_h} L\Big(u_h(\tau), \operatorname{sgn}(u_h'(\tau))\,\rho_h(\tau)\Big)\frac{|u_h'(\tau)|}{\rho_h(\tau)}\,d\tau \\ &\le \frac{1}{\rho_{min}}\int_{A_h} L\Big(u_h(\tau), \operatorname{sgn}(u_h'(\tau))\,\rho_h(\tau)\Big)|u_h'(\tau)|\,d\tau\,. \end{aligned}$$

We thus obtain

$$Q_{2,h} \;\le\; \frac{1}{\rho_{min}}\Big(\sup_{z\in y(I)} L\big(z,\rho^-(z)\big) + \sup_{z\in y(I)} L\big(z,\rho^+(z)\big)\Big)\int_{A_h}|u_h'|\,d\tau$$

so that Condition (R_y) and (4.1) yield that $Q_{2,h} \to 0$ as $h \to +\infty$.
It remains to prove that $Q_{3,h} \to 0$ as $h \to +\infty$. If $T_h \ge 1$, then $Q_{3,h} = 0$. Let us examine the case where $T_h < 1$. From the construction of the extension w_h of v_h on $[T_h, 1]$, we have

$$\begin{aligned} Q_{3,h} \;=\; &\sum_{\substack{0\le i\le m\\ i \text{ even}}} \int_{\tau_i}^{\tau_{i+1}} L\Big(w_h(t), \rho^+(w_h(t))\Big)\,dt \\ &\qquad\qquad + \sum_{\substack{0\le i\le m\\ i \text{ odd}}} \int_{\tau_i}^{\tau_{i+1}} L\Big(w_h(t), \rho^-(w_h(t))\Big)\,dt \end{aligned}$$

so that

$$Q_{3,h} \;\le\; \Big(\sup_{z\in y(I)} L\big(z,\rho^-(z)\big) + \sup_{z\in y(I)} L\big(z,\rho^+(z)\big)\Big)(1-T_h).$$

Since $T_h \to 1$ as $h \to +\infty$, we conclude with the help of Condition (R_y) that $Q_{3,h} \to 0$ as $h \to +\infty$. The proof that $F(w_h) \to F(y)$ as $h \to +\infty$ is now complete. $\square$

Inspecting the proof of Theorem 4.1, we see that we could replace the Lipschitz continuity assumption on the functions ρ^+, ρ^- by the assumption that

$$\sup_{z\in y(I)} L\big(z,0\big) \;<\; +\infty.$$

Indeed, in the case where $T_h < 1$, we could then extend v_h on $[T_h, 1]$ with the help of a constant function on $[T_h, 1]$.

4.2 Non-occurrence of the phenomenon

We give finally a condition ensuring the non–occurrence of the Lavrentiev phenomenon.

Condition (R). There exist two locally Lipschitz functions ρ^-, ρ^+ defined on $\mathbb{R}$ such that:

$$\forall z \in \mathbb{R} \qquad \rho^-(z) < 0\,, \quad \rho^+(z) > 0\,,$$

and for every bounded interval J of $\mathbb{R}$,

$$\sup_{z\in J} L\big(z,\rho^-(z)\big) \;<\; +\infty\,, \quad \sup_{z\in J} L\big(z,\rho^+(z)\big) \;<\; +\infty\,.$$

Corollary 4.5 (**Non-occurrence of the Lavrentiev phenomenon**)**.** Let $L : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ be an autonomous Borel Lagrangian with non–negative values, possibly infinite. Suppose that L satisfies Condition (R). Then the Lavrentiev phenomenon does not occur for (P).

Proof. Let $(y_k)_{k\in\mathbb{N}}$ be a minimizing sequence for (P) satisfying, for all $k \in \mathbb{N}$:

- $y_k \in W^{1,1}(I)$, $y_k(0) = 0$;
- $F(y_k) \le \inf\,(\mathrm{P}) + \dfrac{1}{k+1}$.

Fix $k \in \mathbb{N}$. The condition (R) implies that the condition (R_{y_k}) holds as well. By Theorem 4.1, there exists $z_k \in \mathrm{Lip}([0,1])$ such that $z_k(0) = 0$ and

$$F(z_k) \;\le\; F(y_k) + \frac{1}{k+1}\,.$$

Therefore $(z_k)_{k\in\mathbb{N}}$ is a minimizing sequence of Lipschitz functions for (P), thus proving the claim. □

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