LONG TIME BEHAVIOR FOR A CURVATURE FLOW OF NETWORKS RELATED TO GRAIN BOUNDARY MOTION WITH THE EFFECT OF LATTICE MISORIENTATIONS

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Abstract. The mathematical model of grain boundary motion, including lattice misorientations' effect, is considered. When time-dependent lattice misorientations are state variables of the surface tension of the grain boundary, to ensure the energy dissipation law, one can obtain a curvature flow of networks with time-dependent mobilities. This paper studies the solvability and long-time asymptotic behavior of the curvature flow subjected to the Herring condition which ensures that the constituent grain boundary surface tensions are balanced at the triple junction.

1. Introduction

There are many kinds of research about the curvature flow of networks related to planar grain boundary motion. According to the theory by Mullins and Herring [19, 36, 37], the curvature flow of networks is well-known as a typical model for the evolution of grain boundaries in polycrystalline materials as an evolution law of local interactions. In addition, the classical curvature flow of networks can be derived from the principle of maximum dissipation for the total grain boundaries, under the assumption that the total grain boundary energy depends only on the surface tension of the grain boundaries and the surface tension is constant. When the surface tension depends on the grain boundary's normal vector in order to take into account the grain boundary energy anisotropy, the anisotropic curvature flow of networks can be derived similarly. These flows are well-studied as significant geometric variational problems.

In this paper, we study the case that the grain boundary energy is affected by the shape of the grain boundary and other grain boundary structures. Namely, the surface tension has other state variables in our case. A grain boundary is made from two or more single grains, which have different lattice orientations. A mismatch of the lattice orientations neighboring grains is called a lattice misorientation. Because the lattice orientation is discontinuous at the grain boundary in general, one should include the lattice misorientation in state variables of the grain boundary energy. Kinderlehrer and Liu [25] derived the governed equations, by the principle of maximum dissipation, of the evolution of grain boundaries with the lattice misorientations as a parameter and studied mathematical analysis of the system. Epshteyn, Liu, and the second author [12] developed their work to assume that the lattice misorientations depend on time. Here we briefly explain the case of one triple junction (see figure 1).

We consider the following grain boundary energy of the system

$$E(t) = \sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t), \tag{1.1}$$

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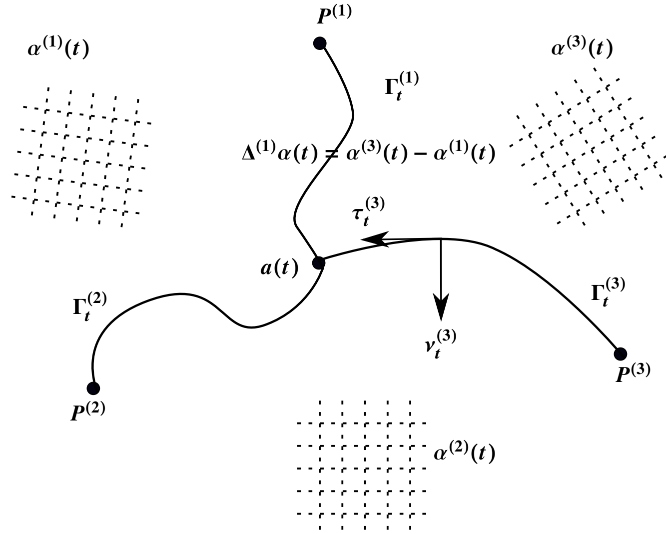

FIGURE 1. Grain boundaries/curves $\Gamma_t^{(j)}$ that meet at a triple junction $\vec{a}(t)$. Lattice orientations are angles (scalars) $\alpha^{(j)}(t)$. Misorientation $\Delta^{(j)}\alpha(t)$ of $\Gamma_t^{(j)}$ is the difference between two lattice orientations of grains that share grain boundary $\Gamma_t^{(j)}$. The unit tangent vector of $\Gamma_t^{(j)}$ with the direction starting from $P_t^{(j)}$ and ending to $\vec{a}(t)$ is denoted by $\tau_t^{(j)}$. The unit normal vector, which is the 90 degree counterclockwise rotation of $\tau_t^{(j)}$, is denoted by $\nu_t^{(j)}$.

where $\sigma = \sigma(\alpha) : \mathbb{R} \to \mathbb{R}$ is a given surface tension of a grain boundary, $\alpha^{(j)} = \alpha^{(j)}(t) : [0, \infty) \to \mathbb{R}$ is a time-dependent orientation of the grains, $\Delta^{(j)}\alpha := \alpha^{(j-1)} - \alpha^{(j)}$ is a lattice misorientation of the grain boundary $\Gamma_t^{(j)}$, and $L^{(j)}(t)$ is the length of $\Gamma_t^{(j)}$ for $j = 1, 2, 3$. For simplicity of notation, we write $\alpha^{(0)} = \alpha^{(3)}$. Impose that three ends of the grain boundaries $\Gamma^{(j)}$ are the same position $\vec{a}(t)$, called a triple junction, and the other end of $\Gamma^{(j)}$ is a given fixed point $P^{(j)}$. In this work, we assume that a grain boundary energy density $\sigma(\Delta^{(j)}\alpha)$ is independent of the normal vector of the grain boundary $\Gamma_t^{(j)}$, namely, our system is isotropic with respect to $\Gamma_t^{(j)}$. As a result of applying the principle of maximum dissipation for the energy (1.1), the following model was derived in [12]:

$$
\begin{cases}
V_t^{(j)} = \mu\sigma(\Delta^{(j)}\alpha)\kappa_t^{(j)}, \quad \text{on } \Gamma_t^{(j)},\ t > 0,\ j = 1, 2, 3,\\
\dfrac{d\alpha^{(j)}}{dt} = -\gamma\Big(\partial_\alpha\sigma(\Delta^{(j+1)}\alpha)L^{(j+1)}(t) - \partial_\alpha\sigma(\Delta^{(j)}\alpha)L^{(j)}(t)\Big), \quad t > 0,\ j = 1, 2, 3,\\
\dfrac{d\vec{a}}{dt}(t) = -\eta\displaystyle\sum_{k=1}^{3}\sigma(\Delta^{(k)}\alpha)\tau_t^{(k)}, \quad t > 0,\ \text{at } \vec{a},\\
\vec{a}(t) = \vec{\xi}^{(1)}(1, t) = \vec{\xi}^{(2)}(1, t) = \vec{\xi}^{(3)}(1, t), \quad \text{and} \quad \vec{\xi}^{(j)}(0, t) = P^{(j)}, \quad t > 0,\ j = 1, 2, 3,
\end{cases}
\tag{1.2}
$$

where $\xi^{(j)}(\cdot, t) : [0, 1] \to \mathbb{R}^2$ is a parametrization of $\Gamma_t^{(j)}$ for $t \geq 0$, $V_t^{(j)}$ is the normal velocity of $\Gamma_t^{(j)}$, $\kappa_t^{(j)}$ is the curvature of $\Gamma_t^{(j)}$, $\mu, \gamma, \eta > 0$ are constants, $\tau_t^{(k)}$ is the unit tangent vector on $\Gamma_t^{(k)}$ and $P^{(j)}$ is a fixed point. For the sake of notational simplicity, we denote $\Delta^{(4)}\alpha = \Delta^{(1)}\alpha$. From (1.2), we have energy dissipation which took a form as presented below;

$$\frac{dE}{dt} = -\sum_{j=1}^{3}\left(\frac{1}{\mu}\int_{\Gamma_t^{(j)}} |V_t^{(j)}|^2 ds + \frac{1}{\gamma}\left|\frac{d\alpha^{(j)}}{dt}(t)\right|^2\right) - \frac{1}{\eta}\left|\frac{d\vec{a}}{dt}\right|^2,$$

where s is the arc-length parameter of $\Gamma_t^{(j)}$. Note that the third equation of (1.2) is a kind of dynamic boundary condition for the differential equations. When we take the relaxation limit $\eta \to \infty$, the third equation turns into a force balance condition, known as the Herring condition, at the triple junctions. More in-depth discussion and complete details of the derivation of the model (1.2) can be found in the recent paper by Epshteyn, Liu and the second author [12, Section 2].

In [11, 12], they relaxed the curvature effect by taking the limit $\mu \to \infty$ and derived an ODE system, and studied well-posedness, long-time asymptotics, and numerical analysis of the system. In [13], they considered ensembles of triple junctions and misorientations (without the curvature effect), and they used white noises to describe interactions among the grain boundaries and the triple junctions in a grain boundary network, including modeling of critical/disappearance events, e.g., grain disappearance, facet/grain boundary disappearance, facet interchange and splitting of unstable junctions. However, the well-posedness and long-time asymptotics of the system with the curvature effect is not still well-known.

On the other hand, the second and third authors [35] considered the well-posedness and long-time asymptotics of a curve shortening equation related to the grain boundary motion (1.2). The curve shortening equation was derived from the principle of maximum dissipation of the energy $\sigma(\Delta\alpha)|\Gamma_t|$ with the periodic boundary condition. Since the misorientation is a state variable of the energy and depends on time, the associated curve shortening equation has a time-dependent mobility coefficient. However, the interaction among the grain boundaries at the triple junction, even though the limiting case $\eta \to \infty$, is not well-known since the prior work studied only one grain boundary.

In this paper, we consider (1.2) in the case $\eta \to \infty$, namely, curvature flow of networks with time dependent mobility governed by the evolving lattice misorientations, and with the Herring condition at the triple junction. We thus consider a motion of connected curves $\Gamma_t^{(j)} \subset \mathbb{R}^2 (j = 1, 2, 3)$ and anisotropy parameters $\alpha^{(j)}(t) \in \mathbb{R}$ governed by

$$V_t^{(j)} = \sigma(\Delta^{(j)}\alpha(t))\kappa_t^{(j)} \quad \text{on } \Gamma_t^{(j)}, \tag{1.3}$$

$$\partial_t \alpha^{(j)}(t) = -\gamma\{\partial_\alpha \sigma(\Delta^{(j+1)}\alpha(t))L^{(j+1)}(t) - \partial_\alpha \sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t)\}, \tag{1.4}$$

$$\vec{a}(t) = \vec{\xi}^{(1)}(1,t) = \vec{\xi}^{(2)}(1,t) = \vec{\xi}^{(3)}(1,t), \tag{1.5}$$

$$\sum_{k=1}^{3} \sigma(\Delta^{(k)}\alpha(t))\tau_t^{(k)} = 0 \quad \text{at } \vec{a}, \tag{1.6}$$

$$\vec{\xi}^{(j)}(0,t) = P^{(j)}. \tag{1.7}$$

Here, we let μ in (1.2) be 1 without loss of generality. Throughout the paper, a unit normal vector $\nu_t^{(j)}$ of $\Gamma_t^{(j)}$ is chosen as the 90 degree rotation of $\tau_t^{(j)}$ counter-clockwise. The velocity $V^{(j)}$ and the curvature $\kappa_t^{(j)}$ are positive if the normal velocity vector and the curvature vector face the same direction of $\nu_t^{(j)}$, respectively. Furthermore, the curves $\Gamma_t^{(j)}$ are numbered counter-clockwise around the junction point. According to the argument as in [6], the Herring condition (1.6) implies that

$$\frac{\sigma(\Delta^{(1)}\alpha(t))}{\sin\theta^{(2)}} = \frac{\sigma(\Delta^{(2)}\alpha(t))}{\sin\theta^{(3)}} = \frac{\sigma(\Delta^{(3)}\alpha(t))}{\sin\theta^{(1)}}$$

by letting $\theta^{(j)}(t) \in (0, 2\pi)$ be the angle between $\Gamma_t^{(j)}$ and $\Gamma_t^{(j+1)}$. Therefore, the angles at the junction change with time evolution in our setting, which is an especially different behavior from the classical curvature flow. Notice that all angles are always 120 agree if σ is constant. We thus intend to study the existence theory and long-time asymptotics of the geometric flow governed by the system (1.3)–(1.7).

For the classical curvature flow of networks, in other word, for the case that σ is a constant, the existence theory was studied by Bronsard-Reitich [6] for initial triod of class $C^{2+\beta}$. Mantegazza-Novaga-Tortorelli [34] extended the result in the case that the initial network has four or more grains of same class. The regularity assumption on the initial network to obtain the uniqueness of the flow was recently improved by [17] in Sobolev classes. In the all articles, they first re-formulate the geometric flow to a system of the parametrization $\xi^{(j)}$ and apply a classical existence theory for the system. Mantegazza-Novaga-Tortorelli [34] also proved that the flow becomes smooth immediately by deriving energy type estimates for the derivatives of the curvatures. In this argument, similar energy type estimates for the tangent velocity is required since the regularity of the flow can be discussed from the smoothness of $\xi^{(j)}$ and the system of $\xi^{(j)}$ depends on not only the normal velocity but also the tangent velocity. We note that the smoothing effect was proved also in [17] introducing a linearized operator around the unique solution and applying functional analysis. In this paper, we re-formulate the geometric flow (1.3)–(1.7) to a system of an angle $\Theta^{(j)}$ of the tangent vector (see Remark 2.6 for the relationship between $\Theta^{(j)}$ and $\xi^{(j)}$), the orientation $\alpha^{(j)}$ and the length of the curves $L^{(j)}$. Since the curvature coincides with the derivative of $\Theta^{(j)}$ with respect to the arc-length, the re-formulation make it easier to discuss the relation between the regularity/asymptotics of $\Theta^{(j)}$ and the curvature. We thus can skip deriving energy type estimates of the tangent velocity (see also Remark 6.15).

First present result is the short time existence of the geometric flow starting from a smooth initial smooth triod under the following assumption.

(A1) $\gamma > 0, \sigma \in C^\infty(\mathbb{R}), \sigma(\alpha) > 0$ for any $\alpha \in \mathbb{R}$.

We apply a classical existence theory for systems as in Ladyženskaja-Solonnikov-Ural'ceva's book [26] to a linearized system of $\Theta^{(j)}$, $\alpha^{(j)}$ and $L^{(j)}$ almost in line with the arguments in [6, 34]. The compatibility condition for the initial curves and the complementing condition for the boundary conditions (see Section 2.2 for the details of the definitions) will be required to apply the theory. We present the existence result as follows.

Theorem 1.1. *Assume (A1). Let $k \in \mathbb{N}$ with $k \geq 3$ and $\beta \in (0, 1)$. Assume that $\Gamma_0^{(j)}$ is a regular $C^{k+\beta}$ curve and connect the fixed point $P^{(j)}$ and a junction point $\vec{a}(0)$ for $j \in \{1, 2, 3\}$. Assume also each angle between $\Gamma_0^{(j+1)}$ and $\Gamma_0^{(j)}$ at the junction point is less than π. Further assume that the pair of $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}} \subset \mathbb{R}^3$ satisfies the compatibility condition of order k for the geometric flow* (1.3)–(1.7)*. Then, there exists $T > 0$ such that a geometric flow governed by* (1.3)–(1.7) *with initial datum $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}}$, whose triod is of class $C^{k+\beta}$ at every time, uniquely exists until the time T.*

Furthermore, if $\Gamma_0^{(j)}$ is smooth for $j \in \{1, 2, 3\}$ and the pair of $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}}$ satisfies the compatibility condition of any order for the geometric flow (1.3)–(1.7)*, then a smooth geometric flow governed by* (1.3)–(1.7) *with initial datum $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}}$ uniquely exists until some time $T' > 0$.*

Note that the compatibility condition at least requires the following two conditions; the initial triod and initial orientation parameters satisfy the Herring condition (1.6) at $t = 0$; and the initial angle condition at the junction point ensures that the linearized system satisfies the complimenting condition. The more detailed regularity of the geometric flow will be stated at the level of parametrization as in Proposition 2.14. The extension of the existence result as in [17] is a future work to obtain the uniqueness of the geometric flow with initial triods of lower regularity classes.

We now focus on the asymptotics of the geometric flow of triods. For the classical curvature flow, Magni, Mantegazza and Novaga [32] proved the L^2-boundedness of the curvatures by combining blow-up arguments and Huisken's monotonicity formula [20]. This boundedness and the energy type estimates in [34] imply the convergence of the flow of triods to the stationary Steiner triod connecting three endpoints $P^{(j)}$ in the C^∞ topology. One of the key properties in this stability result is that the Steiner triod is a unique minimizer of (1.1) with constant σ if the following condition (A2) holds.

(A2) Each interior angle of the triangle generated by the fixed points $\{P^{(j)}\}_{j\in\{1,2,3\}}$ is less than $2\pi/3$.

We note that, in this case, the Steiner triod is the union of three line segments connecting the fixed point $P^{(j)}$ and the Fermat point of the triangle with vertexes $P^{(1)}, P^{(2)}$ and $P^{(3)}$. In our problem, as a first step to analyze the asymptotics, we further assume the following condition (A3), which yields the convexity of σ.

(A3) σ satisfies $\partial_\alpha\sigma(0) = 0$ and $\partial_\alpha^2\sigma(\alpha) > 0$ for any $\alpha \in \mathbb{R}$.

The equilibrium of our problem is then a family of the unique Steiner triod and orientations with 0-misorientation. We note that an equilibrium with non-zero misorientations possibly exists if σ is non-covex (see Remark 3.2). We further note that the monotonicity formula can not be easily extended to our problem because of the differently development of the surface tensions $\sigma(\Delta^{(j)}\alpha)$. Indeed, the monotonicity formula for the flow of just one curve was proved in [35] by applying a time-rescaling along with the development of the surface tension. In our problem, it may be difficult to choose a time-rescaling along with the development of the all surface tensions $\sigma(\Delta^{(j)}\alpha)$. Therefore, the analysis on the asymptotics as in [32] can not be easily applied to our problem. We thus extend the exponential L^2-decay estimate of the curvatures as in [15] for the classical curvature flow of triods with Neumann boundary condition. We also modify the dissipation estimate of the misorientation as in [12] for the system (1.2) with relaxed the curvature effect to obtain the exponential decay of the misorientations in our problem. As a result, extending the energy type estimates as in [34] for our problem, we obtain the smoothing effect, the global existence result and the asymptotic result under a closeness of initial datum to the equilibrium as follows.

Theorem 1.2. *Let $\gamma, \sigma, P^{(j)} (j = 1, 2, 3)$ satisfy the assumptions (A1)–(A3). Assume that each initial regular curve $\Gamma^{(j)}$ is of class H^2 and connects the fixed point $P^{(j)}$ and a junction point $\vec{a}(0)$ for $j \in \{1, 2, 3\}$. Further assume the pair of the initial triod and the initial orientations $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}}$ satisfies the Herring condition (1.6) at $t = 0$. Then, there exist $m > 0$ and $\varepsilon > 0$ such that a geometric flow governed by (1.3)–(1.7) with the initial datum $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}}$, whose triod is smooth at every positive time, exists globally in time if*

$$\sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha_0) L^{(j)}(0) \le \sigma(0) m, \quad \sum_{j=1}^{3} \left\{ \left(\Delta^{(j)}\alpha_0\right)^2 + \int_{\Gamma_0^{(j)}} \left(\sigma(\Delta^{(j)}\alpha_0)\right)^2 (\kappa_0^{(j)})^2 \, ds \right\} \le \varepsilon, \tag{1.8}$$

where s is the arc-length for each initial curve $\Gamma_0^{(j)}$. Furthermore, the triod $\cup_{j=1}^{3}\Gamma_t^{(j)}$ converges exponentially fast to the unique Steiner triod connecting the three fixed endpoints $P^{(j)}$ in the C^∞ topology and $\alpha^{(j)}(t)$ converges exponentially fast to $(\alpha_0^{(1)} + \alpha_0^{(2)} + \alpha_0^{(3)})/3$ as $t \to \infty$ for any $j \in \{1, 2, 3\}$.

Here, we note that the uniqueness of the flow is unknown in our method except the case that the initial triod is of class $C^{3+\beta}$, while we can obtain the Hölder continuity (with respect to the time variable) of the moving triod at the initial time in $C^{1+\beta}$ topology for arbitrary $\beta \in (0, 1/2)$ (see also Corollary 6.14).

We further refer to other works related to our problem. For the classical curvature flow of triods with a homogenous Neumann boundary in bounded domain, the asymptotics was studied in [15, 21]. The former study [21] shows that the linear stability of stationary triod depends on the curvature of the domain at the three end-points of the triod. The latter study [15] gives a proof of the local exponential stability of the stationary triod when it has linear stability. Some arguments in the previous studies are adopted in this paper to derive the Poincaré type inequality, which is a key to prove the exponential L^2-decay of the curvatures. We also note that, for classical curvature flow of networks with more than 3 fixed boundary points, the local asymptotic stability of the minimizer of the total length was recently shown by Pluda-Pozzetta [38] by investigating the Łojasiewicz-Simon inequality for networks. On the other hand, for the interior behavior of networks with a lot of grains, a part of grains may vanish in finite time. The shrinking result was first proved in [14, 18] for motion of Jordan curves and self similar shrinkers were constructed. We also refer to [3, 4, 7] for the examples of the classification of the self similar shrinkers in the motion of networks. Mantegazza-Novaga-Tortorelli [34] also studied type I and type II singularities in addition to the existence theorem and the smoothing effect. Because of the singularities, weak solutions of the multi-phase mean curvature flow have also been studied well. We here refer to [5, 23, 24] for the existence theory of measure theoretic solutions, which are so-called the Brakke flow, and [27, 28, 29] for the construction of distributional solutions in the framework of BV functions. The existence of strong solutions without the Herring condition at the initial time is also important in view of the analysis of the singularities since the Herring condition is lost when a grain vanishes. This kind of the existence theory was studied in [22, 30, 33].

The rest of the paper is organized in the following way. We first introduce a re-formulation of the geometric flow and prove Theorem 1.1 in Section 2. The equilibriums will be studied in Section 3. Section 4 lists some geometric properties for the computations in the sequel. The exponential decay of the misorientations is also prove in this section. The exponential L^2-decay of the curvatures is proved in Section 5 and higher order estimates are derived in Section 6 to continue to prove Theorem 1.2.

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2. Local existence theory in a smooth setting

In this section, we prove the local existence of the geometric flow in a smooth setting. We first introduce an angle function of $\Gamma_t^{(j)}$ and re-formulate the problem to apply a classical theory for parabolic partial differential equations. We note that our re-formulation is different from it in [6, 34] as we mentioned in the introduction.

2.1. **Parametrization of $\Gamma_t^{(j)}$ and re-formulation of the problem.** Let the curve $\Gamma_t^{(j)}$ be parametrized by a smooth map $\xi^{(j)} = \xi^{(j)}(x,t) : [0,1]\times[0,T) \to \mathbb{R}^2$ and define an angle function $\Theta^{(j)} = \Theta^{(j)}(x,t) : [0,1]\times[0,T)$ by

$$\tau_t^{(j)} = \frac{\partial_x \xi^{(j)}}{|\partial_x \xi^{(j)}|} = \begin{pmatrix} \cos\Theta^{(j)} \\ \sin\Theta^{(j)} \end{pmatrix} \tag{2.1}$$

to re-formulate (1.3) and (1.5)–(1.7). Here, the angle Θ is chosen to be continuous with respect to x and t, and to satisfy $\Theta^{(1)}(0,0) \in [0,\pi)$ and $\Theta^{(j+1)}(1,0) - \Theta^{(j)}(1,0) \in [0,2\pi)$. Therefore, the angle $\theta^{(j)}(t)$ between $\Gamma_t^{(j)}$ and $\Gamma_t^{(j+1)}$ at the junction coincides with $\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t)$ modulo 2π. Let $\lambda_t^{(j)}$ be the tangent velocity of $\Gamma_t^{(j)}$. Then we have

$$\partial_t \xi^{(j)} = V_t^{(j)} \nu_t^{(j)} + \lambda_t^{(j)} \tau_t^{(j)} = \sigma(\Delta^{(j)}\alpha(t))\kappa_t^{(j)} \nu_t^{(j)} + \lambda_t^{(j)} \tau_t^{(j)}.$$

Recall that we have rescaled so that $\mu = 1$ for (1.2) when we introduce the geometric flow (1.3)–(1.7). Notice also that the arc-length s of $\Gamma_t^{(j)}$ satisfies

$$\partial_s = \frac{1}{|\partial_x \xi^{(j)}|}\partial_x. \tag{2.2}$$

We discuss some geometric identities to re-formulate the geometric flow.

Lemma 2.1. *Any smooth geometric flow satisfying* (1.3)–(1.7) *fulfills the following identities.*

$$\partial_t|\partial_x\xi^{(j)}| = (-V_t^{(j)}\kappa_t^{(j)} + \partial_s\lambda_t^{(j)})|\partial_x\xi^{(j)}| = (-\sigma(\Delta^{(j)}\alpha)(\partial_s\Theta^{(j)})^2 + \partial_s\lambda_t^{(j)})|\partial_x\xi^{(j)}|, \tag{2.3}$$

$$\partial_t\Theta^{(j)} = \partial_s V_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)} = \sigma(\Delta^{(j)}\alpha)\partial_s^2\Theta^{(j)} + (\partial_s\Theta^{(j)})\lambda_t^{(j)}, \tag{2.4}$$

for any $(x,t) \in [0,1]\times[0,T)$ *and* $j \in \{1,2,3\}$. *Furthermore,*

$$\kappa_t^{(j)} = \partial_s\Theta^{(j)} = \lambda_t^{(j)} = 0 \quad \text{at } P^{(j)} \tag{2.5}$$

for any $j \in \{1,2,3\}$ *and*

$$\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha)V_t^{(j)} = \sum_{j=1}^{3}(\sigma(\Delta^{(j)}\alpha))^2\partial_s\Theta^{(j)} = 0 \quad \text{at } \vec{a}(t) \tag{2.6}$$

for any $j \in \{1,2,3\}$.

Proof. The equalities (2.3) and (2.4) can be obtained by a standard argument, (1.3) and $\kappa_t^{(j)} = \partial_s\Theta^{(j)}$. We thus refer to [8, Chapter 1] for the details of the proof. We now prove only (2.5) and (2.6).

From $\xi^{(j)}(0,t) = P^{(j)}(t)$ for any $t \in [0,T)$ and $j \in \{1,2,3\}$, taking the time derivatives on both sides, we have

$$V_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)} = \partial_t\xi^{(j)}(0,t) = 0$$

at the boundary point. Since $\nu_t^{(j)}$ and $\tau_t^{(j)}$ are linearly independent, we have (2.5).

Since $\xi^{(1)}(1,t) = \xi^{(2)}(1,t) = \xi^{(3)}(1,t)$ at the junction point $\vec{a}(t)$, those time derivatives are also same. We thus have by taking inner product $\partial_t\xi^{(i)}(1,t)$ with (1.6) rotated by 90 degrees counter-clockwise

$$\begin{aligned} 0 &= \left\langle \sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha)\nu_t^{(j)}, \partial_t\xi^{(i)}(1,t)\right\rangle \\ &= \sum_{j=1}^{3}\langle\sigma(\Delta^{(j)}\alpha)\nu_t^{(j)}, V_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)}\rangle = \sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha)V_t^{(j)} \end{aligned}$$

at the junction point $\vec{a}(t)$. □

We will derive a representation formula of $\lambda_t^{(j)}$ and substitute it into (2.4). The following lemma shows a representation formula of $\lambda_t^{(j)}$ at the junction point.

Lemma 2.2. *Any smooth geometric flow satisfying* (1.3)–(1.7) *fulfills*

$$\begin{pmatrix}\lambda_t^{(1)}\\ \lambda_t^{(2)}\\ \lambda_t^{(3)}\end{pmatrix} = -\frac{1}{1-c^{(1)}c^{(2)}c^{(3)}}\begin{pmatrix}c^{(1)}c^{(2)}s^{(3)} & s^{(1)} & c^{(1)}s^{(2)}\\ c^{(2)}s^{(3)} & s^{(1)}c^{(2)}c^{(3)} & s^{(2)}\\ s^{(3)} & s^{(1)}c^{(3)} & c^{(1)}s^{(2)}c^{(3)}\end{pmatrix}\begin{pmatrix}\sigma(\Delta^{(1)}\alpha)\partial_s\Theta^{(1)}\\ \sigma(\Delta^{(2)}\alpha)\partial_s\Theta^{(2)}\\ \sigma(\Delta^{(3)}\alpha)\partial_s\Theta^{(3)}\end{pmatrix} \tag{2.7}$$

at $\vec{a}(t)$ *(or at* $x=1$*), where* $c^{(j)} = \cos(\Theta^{(j+1)} - \Theta^{(j)})$ *and* $s^{(j)} = \sin(\Theta^{(j+1)} - \Theta^{(j)})$ *for* $j = 1, 2, 3$.

Proof. We have by (1.5)

$$\partial_t\vec{a}(t) = V_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)} = V_t^{(j+1)}\nu_t^{(j+1)} + \lambda_t^{(j+1)}\tau_t^{(j+1)},$$

which implies by taking the inner product with $\tau_t^{(j)}$

$$\lambda_t^{(j)} = -s^{(j)}V_t^{(j+1)} + c^{(j)}\lambda_t^{(j+1)}. \tag{2.8}$$

We thus obtain by a simple calculation

$$\begin{pmatrix}\lambda_t^{(1)}\\ \lambda_t^{(2)}\\ \lambda_t^{(3)}\end{pmatrix} = -\frac{1}{1-c^{(1)}c^{(2)}c^{(3)}}\begin{pmatrix}c^{(1)}c^{(2)}s^{(3)} & s^{(1)} & c^{(1)}s^{(2)}\\ c^{(2)}s^{(3)} & s^{(1)}c^{(2)}c^{(3)} & s^{(2)}\\ s^{(3)} & s^{(1)}c^{(3)} & c^{(1)}s^{(2)}c^{(3)}\end{pmatrix}\begin{pmatrix}V_t^{(1)}\\ V_t^{(2)}\\ V_t^{(3)}\end{pmatrix}.$$

Apply (1.3) and $\partial_s\Theta^{(j)} = \kappa_t^{(j)}$ to obtain (2.7). □

Since the tangent velocity $\lambda_t^{(j)}$ depends on the choice of the parametrization of $\Gamma_t^{(j)}$, we thus restrict the parametrization to satisfy

$$|\partial_x\xi^{(j)}(x,t)| = L^{(j)}(t) \quad \text{for } x \in [0,1]. \tag{2.9}$$

A similar restriction can be seen in [9]. Note that we will construct a geometric flow satisfying (2.9) later (see also Remark 2.6). Then, $\lambda_t^{(j)}$ can be determined uniquely and we obtain the following formula.

Lemma 2.3. *For any smooth geometric flow, let* $\xi^{(j)} : [0,1]\times[0,T)$ *be a parametrization of* $\Gamma_t^{(j)}$ *satisfying* (2.9). *Then,*

$$\lambda_t^{(j)}(x) = \frac{\sigma(\Delta^{(j)}\alpha(t))}{L^{(j)}(t)}\int_0^x (\partial_x\Theta^{(j)}(\tilde{x},t))^2\,d\tilde{x} + x\frac{d}{dt}L^{(j)}(t) \tag{2.10}$$

for $(x,t) \in [0,1]\times[0,T)$ *and* $j \in \{1,2,3\}$.

Proof. Taking the time derivative on the both sides of the square of (2.9) and dividing the equality by 2, we have by Frenet-Serret formulas

$$\begin{aligned} L^{(j)}(t)\frac{d}{dt}L^{(j)}(t) &= \langle\partial_x\xi^{(j)}, \partial_x\partial_t\xi^{(j)}\rangle \\ &= \langle\partial_x\xi^{(j)}, \partial_x(V_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)})\rangle \\ &= (L^{(j)}(t))^2\langle\partial_s\xi^{(j)}, \partial_s(V_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)})\rangle \\ &= (L^{(j)}(t))^2(-\kappa_t^{(j)}V_t^{(j)} + \partial_s\lambda_t^{(j)}), \end{aligned}$$

which implies

$$\partial_x \lambda_t^{(j)} = \frac{\sigma(\Delta^{(j)}\alpha(t))}{L^{(j)}(t)}(\partial_x \Theta^{(j)})^2 + \frac{d}{dt}L^{(j)}(t)$$

due to (1.3) and $\partial_s \Theta^{(j)} = \kappa_t^{(j)}$. Integrating it with respect to x and applying (2.5), we have (2.10). □

Plugging (2.10) into (2.4), we obtain a differential equation which involves $L^{(j)}(t)$ in the coefficients. We thus have to derive a differential equation of $L^{(j)}(t)$ to re-formulate the geometric flow into a system of $\Theta^{(j)}, \alpha^{(j)}$ and $L^{(j)}$. We obtain immediately the following equation by substituting $x = 1$ into (2.10).

Lemma 2.4. *Any smooth geometric flow satisfying* (2.9) *fulfills*

$$\frac{d}{dt}L^{(j)}(t) = -\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\int_0^1 (\partial_x \Theta^{(j)})^2\, dx + \lambda_t^{(j)}(1). \tag{2.11}$$

Summarizing the results so far, we obtain the following system. Hereafter, let $\vec{\Theta} = (\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)})$, $\vec{L} = (L^{(1)}, L^{(2)}, L^{(3)})$ and $\vec{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$ for simplicity.

Proposition 2.5. *Any smooth geometric flow satisfying* (1.3)–(1.7) *and* (2.9) *fulfills the following system of* $\vec{\Theta}, \vec{L}$ *and* $\vec{\alpha}$.

$$\begin{cases} \partial_t \Theta^{(j)} = \frac{\sigma(\Delta^{(j)}\alpha)}{(L^{(j)})^2}\partial_x^2 \Theta^{(j)} + f^{(j)}(\vec{\Theta}, \vec{L}, \vec{\alpha}), & (x,t) \in (0,1)\times(0,T), \quad j = 1,2,3, \\ \partial_x \Theta^{(j)}(0,t) = 0, & t \in (0,T), \quad j = 1,2,3, \\ \sum_{j=1}^3 \sigma(\Delta^{(j)}\alpha(t)) \cos \Theta^{(j)}(1,t) = 0, & t \in (0,T), \\ \sum_{j=1}^3 \sigma(\Delta^{(j)}\alpha(t)) \sin \Theta^{(j)}(1,t) = 0, & t \in (0,T), \\ \sum_{j=1}^3 \frac{(\sigma(\Delta^{(j)}\alpha(t)))^2}{L^{(j)}(t)}\partial_x \Theta^{(j)}(1,t) = 0, & t \in (0,T), \\ \partial_t L^{(j)} = -\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\int_0^1 (\partial_x \Theta^{(j)})^2\, dx + g^{(j)}(\vec{\Theta}, \vec{\alpha}), & t \in (0,T), \quad j = 1,2,3, \\ \partial_t \alpha^{(j)} = -\gamma\{\partial_\alpha \sigma(\Delta^{(j+1)}\alpha)L^{(j+1)} - \partial_\alpha \sigma(\Delta^{(j)}\alpha)L^{(j)}\}, & t \in (0,T), \quad j = 1,2,3, \end{cases} \tag{2.12}$$

where

$$f^{(j)}(\vec{\Theta}, \vec{L}, \vec{\alpha})(x,t) = \frac{\sigma(\Delta^{(j)}\alpha)\partial_x \Theta^{(j)}(x,t)}{(L^{(j)})^2}\left\{\int_0^x (\partial_x \Theta^{(j)}(\tilde{x},t))^2\, d\tilde{x} - x\int_0^1 (\partial_x \Theta^{(j)}(x,t))^2\, dx\right\} + \frac{x\partial_x \Theta^{(j)}(x,t)}{L^{(j)}} g^{(j)}(\vec{\Theta}, \vec{\alpha}),$$

$$\begin{aligned} (\lambda_t^{(j)}(1) =) g^{(j)}(\vec{\Theta}, \vec{\alpha})(t) = -\frac{1}{1 - c^{(1)}c^{(2)}c^{(3)}}\Big\{ & c^{(j)}c^{(j+1)}s^{(j+2)}\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\partial_x \Theta^{(j)}(1,t) \\ & + s^{(j)}\frac{\sigma(\Delta^{(j+1)}\alpha)}{L^{(j+1)}}\partial_x \Theta^{(j+1)}(1,t) \\ & + c^{(j)}s^{(j+1)}\frac{\sigma(\Delta^{(j+2)}\alpha)}{L^{(j+2)}}\partial_x \Theta^{(j+2)}(1,t)\Big\}, \end{aligned} \tag{2.13}$$

$$c^{(j)}(t) = \cos(\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t)), \quad s^{(j)}(t) = \sin(\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t)).$$

Proof. We have (2.12) except the third and forth equalities by combining Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4. The remained equalities follow from (1.6) and the definition of $\Theta^{(j)}$. □

Remark 2.6. For any smooth functions $\Theta^{(j)} : [0,1] \times [0,T) \to \mathbb{R}$ and $L^{(j)} : [0,T) \to (0,\infty)$, and also the fixed end-points $P^{(j)}$, we can construct uniquely a moving curve satisfying (2.1) and (2.9) as

$$\xi^{(j)}(x,t) := \left(\int_0^x L^{(j)}(t) \cos \Theta^{(j)}(\tilde{x},t)\, d\tilde{x}, \int_0^x L^{(j)}(t) \sin \Theta^{(j)}(\tilde{x},t)\, d\tilde{x} \right) + P^{(j)},$$

where $\xi^{(j)}$ is the parametrization of the moving curve. Therefore, our aim can be achieved to construct a geometric flow governed by (1.3)–(1.7) from a solution to (2.12) applying the above representation formula of $\xi^{(j)}$.

2.2. **Linearized system.** We introduce a linearized problem of $\vec{\Theta}$ to apply a standard existence theory for the system as in [26]. A solution to (2.12) will be found by constructing a contraction map from the linearized problem and the differential equations of $\vec{L}$ and $\vec{\alpha}$ in (2.12). We thus consider the following linearized problem around initial datum $\vec{\Theta}_0, \vec{L}_0$ and $\vec{\alpha}_0$ of $\vec{\Theta}, \vec{L}$ and $\vec{\alpha}$, respectively, for the differential equation of $\vec{\Theta}$ in (2.12).

$$\begin{cases} \partial_t \Theta^{(j)} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \partial_x^2 \Theta^{(j)} = F^{(j)}(x,t), & (x,t) \in (0,1) \times (0,T),\ j \in \{1,2,3\}, \\ \partial_x \Theta^{(j)}(0,t) = 0, & t \in (0,T), \quad j \in \{1,2,3\} \\ \sum_{j=1}^3 \left(\sigma(\Delta^{(j)}\alpha_0) \sin \Theta_0^{(j)}(1) \right) \Theta^{(j)}(1,t) = b_1(t), & t \in (0,T), \\ \sum_{j=1}^3 \left(\sigma(\Delta^{(j)}\alpha_0) \cos \Theta_0^{(j)}(1) \right) \Theta^{(j)}(1,t) = b_2(t), & t \in (0,T), \\ \sum_{j=1}^3 \frac{(\sigma(\Delta^{(j)}\alpha_0))^2}{L_0^{(j)}} \partial_x \Theta^{(j)}(1,t) = b_3(t), & t \in (0,T), \end{cases} \tag{2.14}$$

where $F^{(j)}$ and b_i are given function for $j = 1,2,3$ and $i = 1,2$. We note that if we choose $F^{(j)}$ and b_i are as

$$\begin{aligned} F^{(j)}(x,t) &= \left(\frac{\sigma(\Delta^{(j)}\alpha)}{(L^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) \partial_x^2 \Theta^{(j)} + f^{(j)}(\vec{\Theta}, \vec{L}, \vec{\alpha}), \\ b_1(t) &= \sum_{j=1}^3 \left\{ \sigma(\Delta^{(j)}\alpha) \cos \Theta^{(j)}(1,t) + \left(\sigma(\Delta^{(j)}\alpha_0) \sin \Theta_0^{(j)}(1) \right) \Theta^{(j)}(1,t) \right\}, \\ b_2(t) &= \sum_{j=1}^3 \left\{ \left(\sigma(\Delta^{(j)}\alpha_0) \cos \Theta_0^{(j)}(1) \right) \Theta^{(j)}(1,t) - \sigma(\Delta^{(j)}\alpha) \sin \Theta^{(j)}(1,t) \right\}, \\ b_3(t) &= \sum_{j=1}^3 \left(\frac{(\sigma(\Delta^{(j)}\alpha_0))^2}{L_0^{(j)}} - \frac{(\sigma(\Delta^{(j)}\alpha))^2}{L^{(j)}(t)} \right) \partial_x \Theta^{(j)}(1,t), \end{aligned} \tag{2.15}$$

then the system of (2.14) the last two equations (2.12) is equivalent to (2.12). We here construct a solution to (2.14) for general functions $F^{(j)}, b_i$ and initial datum $\vec{\Theta}_0, \vec{L}_0$ and $\vec{\alpha}_0$. In order to apply a standard existence theory for systems as in [26], the parabolicity, complementing condition and compatibility condition should be satisfied. For our system (2.14), since the parabolicity obviously holds, we will discuss the other conditions. We refer to [26, p.601, Definition 4] for the detail of the definition of the parabolicity for systems. First, we show that the complementing condition is satisfied (see [26, Section 9 in Chapter VII] for the detail of the complementing condition).

Lemma 2.7. *The boundary conditions in* (2.14) *satisfy the complementing condition if* $\Theta_0^{(j+1)}(1) - \Theta_0^{(j)}(1) \in (0,\pi)$ *for* $j = 1,2,3$.

Proof. We demonstrate according to Bronsard and Reitich [6] (see also Eidelman and Zhitarashu [10]). We will discuss the complementing condition at $x = 1$ since the condition at $x = 0$ is obviously satisfied. To describe the complementing condition at $x = 1$, let $\mathcal{L}(x, t, \partial_x, \partial_t)$ and $\mathcal{B}(1, t, \partial_x, \partial_t)$ be the 3×3 matrix of the differential equation and the boundary conditions at $x = 1$ in (2.14), respectively, i.e.,

$$\mathcal{L}(x, t, \partial_x, \partial_t)\vec{\Theta} = (F^{(1)}, F^{(2)}, F^{(3)})^T, \quad \mathcal{B}(1, t, \partial_x, \partial_t)\vec{\Theta} = (b_1, b_2, b_3)^T,$$

where $\vec{\Theta} = (\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)})^T$. Then, for $p \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(p) \geq 0$,

$$\mathcal{L}(1, t, \partial_x, p) = \begin{pmatrix} p - \frac{\sigma(\Delta^{(1)}\alpha_0)}{(L_0^{(1)})^2}\partial_x^2 & 0 & 0 \\ 0 & p - \frac{\sigma(\Delta^{(2)}\alpha_0)}{(L_0^{(2)})^2}\partial_x^2 & 0 \\ 0 & 0 & p - \frac{\sigma(\Delta^{(3)}\alpha_0)}{(L_0^{(3)})^2}\partial_x^2 \end{pmatrix},$$

$$\mathcal{B}(1, t, \partial_x, p) = \begin{pmatrix} \sigma(\Delta^{(1)}\alpha_0)\sin\Theta_0^{(1)}(1) & \sigma(\Delta^{(2)}\alpha_0)\sin\Theta_0^{(2)}(1) & \sigma(\Delta^{(3)}\alpha_0)\sin\Theta_0^{(3)}(1) \\ \sigma(\Delta^{(1)}\alpha_0)\cos\Theta_0^{(1)}(1) & \sigma(\Delta^{(2)}\alpha_0)\cos\Theta_0^{(2)}(1) & \sigma(\Delta^{(3)}\alpha_0)\cos\Theta_0^{(3)}(1) \\ \frac{(\sigma(\Delta^{(1)}\alpha_0))^2}{L_0^{(1)}}\partial_x & \frac{(\sigma(\Delta^{(2)}\alpha_0))^2}{L_0^{(2)}}\partial_x & \frac{(\sigma(\Delta^{(3)}\alpha_0))^2}{L_0^{(3)}}\partial_x \end{pmatrix}.$$

Let z be the distance parameter from the boundary $x = 1$ directing the interior of $[0, 1]$. Then we have $\partial_x = -\partial_z$. A general solution of $\mathcal{L}(1, t, -\partial_z, p)W(z; p) = 0$ for $z \in [0, \infty)$ satisfying $|W(z; p)| \to 0$ as $z \to \infty$ is

$$W(z; p) = \left(a_1 e^{-L_0^{(1)}\sqrt{\frac{p}{\sigma(\Delta^{(1)}\alpha_0)}}z}, a_2 e^{-L_0^{(2)}\sqrt{\frac{p}{\sigma(\Delta^{(2)}\alpha_0)}}z}, a_3 e^{-L_0^{(3)}\sqrt{\frac{p}{\sigma(\Delta^{(3)}\alpha_0)}}z}\right)^T$$

for any $\vec{a} = (a_1, a_2, a_3) \in \mathbb{C}^3$, where we choose the square roots so that the real part of them is positive. Let a 3×3 matrix $\mathcal{D}(p)$ be defined by

$$\mathcal{D}(p)\vec{a}^T = \left(\mathcal{B}(1, t, -\partial_z, p)W(z; p)\Big|_{z=0}\right).$$

Applying [10, Lemma I.1], the complementing condition is satisfied if $\det\mathcal{D}(p) \neq 0$ for $p \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(p) \geq 0$. By a simple calculation, we can see that

$$\begin{aligned} \det\mathcal{D}(p) &= \sqrt{p}\left(\prod_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)\right)\det\begin{pmatrix} \sin\Theta_0^{(1)}(1) & \sin\Theta_0^{(2)}(1) & \sin\Theta_0^{(3)}(1) \\ \cos\Theta_0^{(1)}(1) & \cos\Theta_0^{(2)}(1) & \cos\Theta_0^{(3)}(1) \\ \sqrt{\sigma(\Delta^{(1)}\alpha_0)} & \sqrt{\sigma(\Delta^{(2)}\alpha_0)} & \sqrt{\sigma(\Delta^{(3)}\alpha_0)} \end{pmatrix} \\ &= -\sqrt{p}\left(\prod_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)\right)\left(\sum_{j=1}^{3}\sqrt{\sigma(\Delta^{(j+2)}\alpha_0)}\sin\left(\Theta_0^{(j+1)}(1) - \Theta_0^{(j)}(1)\right)\right) \neq 0 \end{aligned}$$

due to the positivity of σ and the assumption $\Theta_0^{(j+1)}(1) - \Theta_0^{(j)}(1) \in (0, \pi)$ for $j = 1, 2, 3$. □

We next discuss the compatibility conditions for the system (2.14). The conditions are related to the regularity of the solution, we thus define the following standard Hölder spaces and the norms.

Definition 2.8. Let I and J be open intervals in $\mathbb{R}$ and $l \in (0, \infty) \setminus \mathbb{N}$. We denote

$$C^l(\bar{I}) = \{f \in C^{[l]}(\bar{I}) \mid \|f\|_{C^l(\bar{I})} < \infty\},$$

where

$$\|f\|_{C^l(\bar{I})} = \|f\|_{C^{[l]}(\bar{I})} + \sup_{x,y \in I, x \neq y}\frac{|\partial_x^{[l]}f(x) - \partial_x^{[l]}f(y)|}{|x - y|^{l-[l]}}.$$

For simplicity, we denote $C^l([0,1])$ and $C^l([0,T])$ by C^l_x and C^l_t, respectively. In addition, we define

$$C^{l,\frac{l}{2}}(\overline{I\times J})=\{f\in C(\overline{I\times J})\mid \partial_t^r\partial_x^s f\in C(\overline{I\times J}) \text{ if } 2r+s<l,\ \|f\|_{C^{l,\frac{l}{2}}(\overline{I\times J})}<\infty\},$$

where

$$\begin{aligned}\|f\|_{C^{l,\frac{l}{2}}(\overline{I\times J})}=&\sum_{j=0}^{[l]}\sum_{2r+s=j}\|\partial_t^r\partial_x^s f\|_{C(\overline{I\times J})}+\sum_{2r+s=[l]}\sup_{x,y\in I,t\in J,x\neq y}\frac{|\partial_t^r\partial_x^s f(x,t)-\partial_t^r\partial_x^s f(y,t)|}{|x-y|^{l-[l]}}\\&+\sum_{0<(l-2r-s)/2<1}\sup_{x\in I,t,s\in J,t\neq s}\frac{|\partial_t^r\partial_x^s f(x,t)-\partial_t^r\partial_x^s f(x,s)|}{|t-s|^{(l-2r-s)/2}}.\end{aligned}$$

As before, we denote $C^{l,\frac{l}{2}}([0,1]\times[0,T])$ by $C^{l,\frac{l}{2}}_{x,t}$, for simplicity.

Definition 2.9. Let functions $F^{(j)}, b_1, b_2$ and b_3 in (2.14) be smooth. Let also a solution $\vec{\Theta}$ to (2.14) starting from $\vec{\Theta}_0$ be of class $(C^{2+\beta,\frac{2+\beta}{2}}_{x,t}([0,1]\times[0,T]))^3$. Then, due to (2.14), the solution should satisfy

$$\partial_x\Theta_0^{(j)}(0)=0, \tag{2.16}$$

$$b_1(0)-\sum_{j=1}^{3}\left(\sigma(\Delta^{(j)}\alpha_0)\sin\Theta_0^{(j)}(1)\right)\Theta_0^{(j)}(1)=0, \tag{2.17}$$

$$b_2(0)-\sum_{j=1}^{3}\left(\sigma(\Delta^{(j)}\alpha_0)\cos\Theta_0^{(j)}(1)\right)\Theta_0^{(j)}(1)=0, \tag{2.18}$$

$$b_3(0)-\sum_{j=1}^{3}\frac{(\sigma(\Delta^{(j)}\alpha_0))^2}{L_0^{(j)}}\partial_x\Theta_0^{(j)}(1)=0 \tag{2.19}$$

and

$$\begin{aligned}0&=\partial_t\left(b_1(t)-\sum_{j=1}^{3}\left(\sigma(\Delta^{(j)}\alpha_0)\sin\Theta_0^{(j)}(1)\right)\Theta^{(j)}(1,t)\right)\Big|_{t=0}\\&=\partial_t b_1(0)-\sum_{j=1}^{3}\left(\sigma(\Delta^{(j)}\alpha_0)\sin\Theta_0^{(j)}(1)\right)\left(\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x^2\Theta_0^{(j)}(1)+F^{(j)}(1,0)\right).\end{aligned} \tag{2.20}$$

We have similarly

$$F^{(j)}(0,0)-\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x\Theta_0^{(j)}(0)=0, \tag{2.21}$$

$$\partial_t b_2(0)-\sum_{j=1}^{3}\left(\sigma(\Delta^{(j)}\alpha_0)\cos\Theta_0^{(j)}(1)\right)\left(\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x^2\Theta_0^{(j)}(1)+F^{(j)}(1,0)\right)=0. \tag{2.22}$$

These equalities can be regarded as boundary conditions of the initial data $\vec{\Theta}_0$. Notice that the boundary conditions in (2.14) also should be satisfied at $t=0$. We say that an initial data $\vec{\Theta}_0\in(C^{2+\beta}_x([0,1]))^3$ satisfies the compatibility condition of order $k\in\mathbb{N}$ for (2.14) if the initial data $\vec{\Theta}_0$ fulfills all boundary conditions of $\vec{\Theta}_0$ which can be derived from a solution $\vec{\Theta}\in(C^{k+\beta,\frac{k+\beta}{2}}_{x,t}([0,1]\times$

$[0,T]))^3$ to (2.14) starting from $\vec{\Theta}_0$ by combining equations in (2.14) and those time derivatives. Note that the formulas (2.16)–(2.22) are the compatibility condition of order 2. The compatibility conditions for (2.12) and the geometric flow (1.3)–(1.7) are similarly defined.

We then obtain the existence theorem for (2.14) due to [26, Theorem 10.1 in Chapter VII].

Proposition 2.10. *Let $T > 0$, $\beta \in (0,1)$ and $k \in \mathbb{N}$ with $k \geq 2$. Assume that σ is a positive smooth function and $\alpha_0^{(j)} \in \mathbb{R}$ and $L_0^{(j)} > 0$ are given constants for $j \in \{1,2,3\}$. Let $F^{(j)} \in C_{x,t}^{(k-2)+\beta, \frac{(k-2)+\beta}{2}}([0,1]\times[0,T])$ for $j \in \{1,2,3\}$, $b_1, b_2 \in C^{\frac{k+\beta}{2}}([0,T])$ and $b_3 \in C_t^{\frac{(k-1)+\beta}{2}}([0,T])$. Assume also $\vec{\Theta}_0 = (\Theta_0^{(1)}, \Theta_0^{(2)}, \Theta_0^{(3)}) \in (C_x^{k+\beta}([0,1]))^3$ satisfy the compatibility condition of order k for (2.14) and $\Theta_0^{(j+1)}(1) - \Theta_0^{(j)}(1) \in (0,\pi)$ for $j = 1,2,3$. Then, there exists $c_1 > 0$ such that (2.14) has a unique solution $\vec{\Theta} = (\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)}) \in (C_{x,t}^{k+\beta, \frac{k+\beta}{2}}([0,1]\times[0,T]))^3$ with $\vec{\Theta}(\cdot,0) = \vec{\Theta}_0$ and the following estimate holds.*

$$\begin{aligned}\sum_{j=1}^{3} \|\Theta^{(j)}\|_{C_{x,t}^{k+\beta,\frac{k+\beta}{2}}} \leq c_1 \Bigg(\sum_{j=1}^{3} \Big(\|F^{(j)}\|_{C_{x,t}^{(k-2)+\beta,\frac{(k-2)+\beta}{2}}} + \|\Theta_0^{(j)}\|_{C_x^{k+\beta}} \Big) \\ + \|b_1\|_{C_t^{\frac{k+\beta}{2}}} + \|b_2\|_{C_t^{\frac{k+\beta}{2}}} + \|b_3\|_{C_t^{\frac{(k-1)+\beta}{2}}} \Bigg).\end{aligned} \tag{2.23}$$

2.3. **Short-time existence.** In this section, we will construct a solution to (2.12) starting from an initial data $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ by applying Proposition 2.10. In the construction of the solution, we have to introduce compatibility conditions for the following reasons, which are different from the compatibility conditions defined in Definition 2.9. Our purpose here is to construct a map

$$(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) \mapsto (\vec{\Theta}, \vec{L}, \vec{\alpha}),$$

where $\underline{\vec{\Theta}} = \underline{\vec{\Theta}}(x,t) : [0,1]\times[0,T] \to \mathbb{R}$, $\underline{\vec{L}} = \underline{\vec{L}}(t) : [0,T] \to \mathbb{R}$ and $\underline{\vec{\alpha}} = \underline{\vec{\alpha}}(t) : [0,T] \to \mathbb{R}$ are functions and $(\vec{\Theta}, \vec{L}, \vec{\alpha})$ is a solution to (2.14) with $F^{(j)}$ and b_j defined by (2.15) with $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$ instead of $(\vec{\Theta}, \vec{L}, \vec{\alpha})$ (see also (2.27) for the form of $F^{(j)}$ and b_j), and to prove that the map is a contraction map. As a result, we can obtain a solution to (2.12) as the fixed point of the contraction map. To construct this map, the initial data $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ needs to satisfy the compatibility condition for (2.14) with $F^{(j)}$ and b_j chosen above. On the other hand, we consider to solve the initial value problem for (2.12) starting from the initial data $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$, and thus we have to fix the initial data. Since $F^{(j)}$ and b_j are constructed from $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$, for the given initial data $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$, we must choose suitable functions $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$ so that the given initial data satisfies the compatibility conditions for (2.14). New compatibility conditions below are related to the suitability of $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$. We note that, in this paper, the former compatibility conditions are defined for the initial data $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ and the new compatibility conditions will be defined for general functions $\underline{\vec{\Theta}}, \underline{\vec{L}}$ and $\underline{\vec{\alpha}}$.

We now introduce the new compatibility conditions. Let $k \in \mathbb{N}$ with $k \geq 2$ and $\beta \in (0,1)$. If $\vec{\Theta} \in (C_{x,t}^{k+\beta,\frac{k+\beta}{2}}([0,1]\times[0,T]))^3$, $\vec{L} \in (C_t^{\frac{k+\beta}{2}}([0,T]))^3$ and $\vec{\alpha} \in (C_t^{\frac{k+\beta}{2}}([0,T]))^3$ are solution to (2.12) starting from initial datum $\vec{\Theta}_0, \vec{L}_0$ and $\vec{\alpha}_0$, then $\partial_t^i \Theta^{(j)}(\cdot,0)$, $\partial_t^i L^{(j)}(0)$ and $\partial_t^i \alpha^{(j)}(0)$ can be represented by the initial datum and the derivatives of $\vec{\Theta}_0$ for $j = 1,2,3$ and $i = 1,2,\cdots,[\frac{k}{2}]$ due to the first and last two equalities in (2.12). For example, if $k = 2$, the representations of $\partial_t \Theta^{(j)}(x,0)$, $\partial_t L^{(j)}(0)$ and

$\partial_t \alpha^{(j)}(0)$ are

$$
\begin{aligned}
&\partial_t \Theta^{(j)}(x,0) = \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x^2 \Theta_0^{(j)}(x) + f^{(j)}(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)(x),\\
&\partial_t L^{(j)}(0) = -\frac{\sigma(\Delta^{(j)}\alpha_0)}{L_0^{(j)}}\int_0^1 (\partial_x \Theta_0^{(j)})^2\, dx + g^{(j)}(\vec{\Theta}_0, \vec{\alpha}_0),\\
&\partial_t \alpha^{(j)}(0) = -\gamma\{\partial_\alpha \sigma(\Delta^{(j+1)}\alpha_0)L_0^{(j+1)} - \partial_\alpha \sigma(\Delta^{(j)}\alpha_0)L_0^{(j)}\}
\end{aligned}
\tag{2.24}
$$

which can be obtained by substituting $t = 0$ into the first and last two equalities in (2.12). As a higher order example, we calculate only the representation of $\partial_t^2 \alpha^{(j)}(0)$. Due to the last two equations in (2.12), we have

$$
\begin{aligned}
\partial_t^2 \alpha^{(j)}(0) &= \partial_t \left(-\gamma\{\partial_\alpha \sigma(\Delta^{(j+1)}\alpha(t))L^{(j+1)}(t) - \partial_\alpha \sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t)\}\right)\Big|_{t=0}\\
&= -\gamma\Big\{\partial_\alpha^2 \sigma(\Delta^{(j+1)}\alpha(t))L^{(j+1)}(t)\big(\partial_t \alpha^{(j)}(t) - \partial_t \alpha^{(j+1)}(t)\big) + \partial_\alpha \sigma(\Delta^{(j+1)}\alpha(t))\partial_t L^{(j+1)}(t)\\
&\qquad - \partial_\alpha^2 \sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t)\big(\partial_t \alpha^{(j-1)}(t) - \partial_t \alpha^{(j)}(t)\big) - \partial_\alpha \sigma(\Delta^{(j)}\alpha(t))\partial_t L^{(j)}(t)\Big\}\Big|_{t=0}\\
&= -\gamma\Big\{\partial_\alpha^2 \sigma(\Delta^{(j+1)}\alpha_0)L_0^{(j+1)}\\
&\qquad\quad \cdot\Big(\gamma\big(\partial_\alpha \sigma(\Delta^{(j+1)}\alpha_0)L_0^{(j+1)} - 2\partial_\alpha \sigma(\Delta^{(j)}\alpha_0)L_0^{(j)} + \partial_\alpha \sigma(\Delta^{(j-1)}\alpha_0)L_0^{(j-1)}\big)\Big)\\
&\qquad - \partial_\alpha \sigma(\Delta^{(j+1)}\alpha_0)\frac{\sigma(\Delta^{(j+1)}\alpha_0)}{L_0^{(j+1)}}\int_0^1 (\partial_x \Theta_0^{(j+1)}(x))^2\, dx + g^{(j+1)}(\vec{\Theta}_0, \vec{\alpha}_0)\\
&\qquad - \partial_\alpha^2 \sigma(\Delta^{(j)}\alpha_0)L^{(j)}\\
&\qquad\quad \cdot\Big(\gamma\big(\partial_\alpha \sigma(\Delta^{(j)}\alpha_0)L_0^{(j)} - 2\partial_\alpha \sigma(\Delta^{(j-1)}\alpha_0)L_0^{(j-1)} + \partial_\alpha \sigma(\Delta^{(j+1)}\alpha_0)L_0^{(j+1)}\big)\Big)\\
&\qquad + \partial_\alpha \sigma(\Delta^{(j)}\alpha_0)\frac{\sigma(\Delta^{(j)}\alpha_0)}{L_0^{(j)}}\int_0^1 (\partial_x \Theta_0^{(j)}(x))^2\, dx + g^{(j)}(\vec{\Theta}_0, \vec{\alpha}_0)\Big\}.
\end{aligned}
$$

We say that general functions, which are not necessary to satisfy any differential equations, $\underline{\vec{\Theta}} \in (C_{x,t}^{k+\beta,\frac{k+\beta}{2}}([0,1]\times[0,T']))^3$, $\underline{\vec{L}} \in (C_t^{\frac{k+\beta}{2}}([0,T']))^3$ and $\underline{\vec{\alpha}} \in (C_t^{\frac{k+\beta}{2}}([0,T']))^3$ satisfy a compatibility condition of order k for (2.12) at the initial time with initial datum $\vec{\Theta}_0 \in (C_x^{k+\beta}([0,1]))^3, \vec{L}_0 \in \{(0,\infty)\}^3$ and $\vec{\alpha}_0 \in \mathbb{R}^3$ if $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})|_{t=0} = (\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ holds and the derivatives $\partial_t^i \underline{\Theta}^{(j)}(\cdot,0)$, $\partial_t^i \underline{L}^{(j)}(0)$ and $\partial_t^i \underline{\alpha}^{(j)}(0)$ satisfy the representations of $\partial_t^i \Theta^{(j)}(\cdot,0)$, $\partial_t^i L^{(j)}(0)$ and $\partial_t^i \alpha^{(j)}(0)$ by the initial datum for $j = 1,2,3$ and $i = 1,2,\cdots,[\frac{k}{2}]$, which follow from the first and last two equalities in (2.12) as above (for example, functions $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$ satisfy the compatibility condition of order 2 for (2.12) if the functions satisfy $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})|_{t=0} = (\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ and (2.24) replaced $\partial_t \Theta^{(j)}(x,0)$, $\partial_t L^{(j)}(0)$ and $\partial_t \alpha^{(j)}(0)$ by $\partial_t \underline{\Theta}^{(j)}(x,0)$, $\partial_t \underline{L}^{(j)}(0)$ and $\partial_t \underline{\alpha}^{(j)}(0)$, respectively).

We next discuss the relation between the former compatibility conditions and the new compatibility conditions.

Lemma 2.11. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\beta, \beta' \in (0,1)$ with $0 < \beta < \beta' < 1$. Assume $\vec{\Theta}_0 \in (C_x^{k+\beta}([0,1]))^3, \vec{L}_0 \in \{(0,\infty)\}^3$ and $\vec{\alpha}_0 \in \mathbb{R}^3$ satisfy the compatibility condition of order k for (2.12).*

Let

$$m_1 := \max_{j=1,2,3} \|\Theta_0^{(j)}\|_{C_x^{k+\beta}([0,1])} + \max_{j=1,2,3} L_0^{(j)} + \max_{j=1,2,3} |\alpha_0^{(j)}|, \quad m_2 := \min_{j=1,2,3} L_0^{(j)}. \tag{2.25}$$

Then, there exists $c_2 > 0$ *depending only on* $\gamma, \sigma, k, \beta, m_1$ *and* m_2 *such that*

$$\begin{aligned} X_T^M := \{(\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}) \in (C_{x,t}^{k+\beta, \frac{k+\beta}{2}}([0,1]\times[0,T]))^3 \times (C_t^{\frac{k+\beta'}{2}}([0,T]))^3 \times (C_t^{\frac{k+\beta'}{2}}([0,T]))^3 : \\ \vec{\underline{\Theta}}, \vec{\underline{L}} \text{ and } \vec{\underline{\alpha}} \text{ satisfy conditions (i) and (ii)}\}, \end{aligned} \tag{2.26}$$

where

(i) $\max_{j=1,2,3} \|\underline{\Theta}^{(j)}\|_{C_{x,t}^{k+\beta}} + \max_{j=1,2,3} \|\underline{L}^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}} + \max_{j=1,2,3} \|\underline{\alpha}^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}} \le M$ *and* $\min_{j=1,2,3,\ t\in[0,T]} \underline{L}^{(j)} \ge m_2/2$

(ii) $(\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}})$ *satisfies the compatibility condition of order* k *for* (2.12) *at the initial time with* $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$,

is non-empty for any $T > 0$ *and* $M > c_2$. *Furthermore, for any* $(\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}) \in X_T^M$, *the initial data* $\vec{\Theta}_0$ *satisfies the compatibility condition of order* k *for* (2.14) *with*

$$\begin{aligned} F^{(j)} &= F^{(j)}(x,t;\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}) = \left(\frac{\sigma(\Delta^{(j)}\underline{\alpha})}{(\underline{L}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) \partial_x^2 \underline{\Theta}^{(j)} + f^{(j)}(\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}), \\ b_1 &= b_1(t;\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}) = \sum_{j=1}^3 \left\{ \sigma(\Delta^{(j)}\underline{\alpha}) \cos \underline{\Theta}^{(j)}(1,t) + \left(\sigma(\Delta^{(j)}\alpha_0) \sin \Theta_0^{(j)}(1) \right) \underline{\Theta}^{(j)}(1,t) \right\}, \\ b_2 &= b_2(t;\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}) = \sum_{j=1}^3 \left\{ \left(\sigma(\Delta^{(j)}\alpha_0) \cos \Theta_0^{(j)}(1) \right) \underline{\Theta}^{(j)}(1,t) - \sigma(\Delta^{(j)}\underline{\alpha}) \sin \underline{\Theta}^{(j)}(1,t) \right\}, \\ b_3 &= b_3(t;\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}}) = \sum_{j=1}^3 \left(\frac{(\sigma(\Delta^{(j)}\alpha_0))^2}{L_0^{(j)}} - \frac{(\sigma(\Delta^{(j)}\underline{\alpha}))^2}{\underline{L}^{(j)}(t)} \right) \partial_x \underline{\Theta}^{(j)}(1,t), \end{aligned} \tag{2.27}$$

where $f^{(j)}$ *is the function defined by* (2.13).

Proof. We construct a typical element of X_T^M when $k = 2$ since a similar argument works even if $k > 2$. Let expand $\Theta_0^{(j)}$ on $\mathbb{R}$ so that $\Theta_0^{(j)} \in C_x^{2+\beta}(\mathbb{R})$ in the following argument and denoted by $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon$ the mollifier with a positive parameter $\varepsilon > 0$. Fix sufficiently small δ. Letting

$$\begin{aligned} \underline{\Theta}^{(j)}(x,t) &:= \Theta_0^{(j)}(x) + \int_0^t \rho_{\tilde{t}^{\frac{1}{2}-\delta}} * \left(\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \partial_x^2 \Theta_0^{(j)} + f^{(j)}(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0) \right)(x)\, d\tilde{t}, \\ \underline{L}^{(j)}(t) &:= L_0^{(j)} + \left(-\frac{\sigma(\Delta^{(j)}\alpha_0)}{L_0^{(j)}} \int_0^1 (\partial_x \Theta_0^{(j)})^2\, dx + g^{(j)}(\vec{\Theta}_0, \vec{\alpha}_0) \right) t, \\ \underline{\alpha}^{(j)}(t) &:= \alpha_0^{(j)} - \gamma \left(\partial_\alpha \sigma(\Delta^{(j+1)}\alpha_0) L_0^{(j+1)} - \partial_\alpha \sigma(\Delta^{(j)}\alpha_0) L_0^{(j)} \right) t \end{aligned}$$

for sufficiently small $t > 0$, we then see that $(\vec{\underline{\Theta}}, \vec{\underline{L}}, \vec{\underline{\alpha}})$ satisfies the compatibility condition of order 2 at the initial time for (2.12). Note that, due to the choice of the parameter of the mollifier, we can see $\underline{\Theta}^{(j)} \in C_{x,t}^{2+\beta, \frac{2+\beta}{2}}([0,1]\times[0,\tilde{T}])$ for sufficient small $\tilde{T} > 0$. For example, the Hölder continuity of

$\partial_x^2\underline{\Theta}^{(j)}(\cdot,t)$ can be obtained as

$$\begin{aligned}\|\partial_x^2\underline{\Theta}^{(j)}(\cdot,t)\|_{C_x^\beta} &\le \|\Theta_0^{(j)}\|_{C_x^{2+\beta}} \\ &\quad + \int_0^t \frac{1}{\tilde{t}^{1-2\delta}}\int_{\mathbb{R}} |\partial_x^2\rho(x)|\,dxd\tilde{t}\left\|\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x^2\Theta_0^{(j)} + f^{(j)}(\vec{\Theta}_0,\vec{L}_0,\vec{\alpha}_0)\right\|_{C_x^\beta} \\ &\le \|\Theta_0^{(j)}\|_{C_x^{2+\beta}} + C\left\|\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x^2\Theta_0^{(j)} + f^{(j)}(\vec{\Theta}_0,\vec{L}_0,\vec{\alpha}_0)\right\|_{C_x^\beta} t^{2\delta}\end{aligned}$$

for small $t>0$ and some constant C depending only on δ and ρ. Therefore, choosing c_2 large enough and adjusting $(\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}})$ away from $t=0$ to satisfy the condition (i), we see that X_T^M is non-empty for any $M>c_2$ and $T>0$.

The remained statement can be seen easily from the definition of the two kind of compatibility conditions. Indeed, since $(\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}}) \in X_T^M$ satisfies the compatibility condition of order 2, the conditions $(\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}})|_{t=0} = (\vec{\Theta}_0,\vec{L}_0,\vec{\alpha}_0)$ and (2.24) replaced $\partial_t\vec{\Theta}(\cdot,0),\partial_t\vec{L}(0),\partial_t\vec{\alpha}(0)$ by $\partial_t\underline{\vec{\Theta}}(\cdot,0),\partial_t\underline{\vec{L}}(0),\partial_t\underline{\vec{\alpha}}(0)$ can be applied to obtain

$$F^{(j)}(1,0;\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}}) = \frac{\partial_x\Theta_0^{(j)}(1)}{L_0^{(j)}}g^{(j)}(\vec{\Theta}_0,\vec{\alpha}_0), \tag{2.28}$$

$$\partial_t b_1(0;\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}}) = -\gamma\sum_{j=1}^{3}\partial_\alpha\sigma(\Delta^{(j)}\alpha_0)(\partial_\alpha\sigma(\Delta^{(j+1)}\alpha_0)L_0^{(j+1)} - \partial_\alpha\sigma(\Delta^{(j)}\alpha_0)L_0^{(j)})\cos\Theta_0^{(j)}(1), \tag{2.29}$$

where $g^{(j)}$ is the function defined by (2.13). On the other hand, since the initial data $(\vec{\Theta}_0,\vec{L}_0,\vec{\alpha}_0)$ satisfies the compatibility condition of order 2 for (2.12), one identity of the compatibility condition

$$\begin{aligned}\sum_{j=1}^{3}\Bigg\{ &-\left(\sigma(\Delta^{(j)}\alpha_0)\sin\Theta_0^{(j)}\right)\left(\frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2}\partial_x^2\Theta_0^{(j)}(1) + \frac{\partial_x\Theta_0^{(j)}(1)}{L_0^{(j)}}g^{(j)}(\vec{\Theta}_0,\vec{\alpha}_0)\right) \\ &+\gamma\partial_\alpha\sigma(\Delta^{(j)}\alpha_0)\left(\partial_\alpha\sigma(\Delta^{(j+1)}\alpha_0)L_0^{(j+1)} + \partial_\alpha\sigma(\Delta^{(j-1)}\alpha_0)L_0^{(j-1)} - 2\partial_\alpha\sigma(\Delta^{(j)}\alpha_0)L_0^{(j)}\right)\cos\Theta_0^{(j)}(1)\Bigg\} = 0\end{aligned}$$

holds. By substituting (2.28) and (2.29) into it, we have (2.20) if $F^{(j)}$ and b_j are chosen as in (2.27). The other identities (2.16)–(2.19), (2.21) and (2.22) can be obtained similarly if $F^{(j)}$ and b_j are chosen as in (2.27). Therefore, the initial data $\vec{\Theta}_0$ satisfies the compatibility condition of order 2 for (2.14) if $F^{(j)}$ and b_j are chosen as in (2.27). When the initial data $(\vec{\Theta}_0,\vec{L}_0,\vec{\alpha}_0)$ satisfies the compatibility condition of order k for (2.12), we can similarly prove that $\vec{\Theta}_0$ satisfies also the compatibility condition of order k for (2.14) with the choice (2.27). □

We next construct a map defined on X_T^M.

Definition 2.12. Let $k\in\mathbb{N}$ with $k\ge 2$ and $\beta,\beta'\in(0,1)$ with $0<\beta<\beta'<1$. Assume $\vec{\Theta}_0 \in (C_x^{k+\beta}([0,1]))^3$, $\vec{L}_0\in\{(0,\infty)\}^3$ and $\vec{\alpha}_0\in\mathbb{R}^3$ satisfy the compatibility condition of order k for (2.12) and $\Theta_0^{(j+1)}(1)-\Theta_0^{(j)}(1)\in(0,\pi)$ for $j=1,2,3$. Let c_2 be the constant in Lemma 2.11. Define X_T and

X_T^M by

$$X_T := \{(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) \in (C_{x,t}^{k+\beta,\frac{k+\beta}{2}}([0,1]\times[0,T]))^3 \times (C_t^{\frac{k+\beta'}{2}}([0,T]))^3 \times (C_t^{\frac{k+\beta'}{2}}([0,T]))^3 : \\ \underline{\vec{\Theta}}(\cdot,0) = \vec{\Theta}_0,\ \ \underline{\vec{L}}(0) = \vec{L}_0,\ \ \underline{\vec{\alpha}}(0) = \vec{\alpha}_0\}$$

and (2.26), respectively, for any $T > 0$ and $M > c_2$. Define also a map $\Phi : X_T^M \to X_T$ by, for $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) \in X_T^M$,

$$\Phi(x,t;\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) := \left(\vec{\Theta}(x,t;\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}), \vec{L}(t;\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}), \vec{\alpha}(t;\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})\right),$$

where $\vec{\Theta}$ is the solution to (2.14) with (2.27) obtained by Proposition 2.10, and $\vec{L}, \vec{\alpha}$ are defined by

$$L^{(j)}(t;\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) := L_0^{(j)} + \int_0^t \left(g^{(j)}(\underline{\vec{\Theta}}, \underline{\vec{\alpha}})(\tilde{t}) - \frac{\sigma(\Delta^{(j)}\underline{\alpha}(\tilde{t}))}{\underline{L}^{(j)}(\tilde{t})} \int_0^1 (\partial_x \underline{\Theta}^{(j)}(x,\tilde{t}))^2\, dx \right) d\tilde{t}$$

$$\alpha^{(j)}(t;\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) = \alpha_0^{(j)} - \gamma \int_0^t \left(\partial_\alpha \sigma(\Delta^{(j+1)}\underline{\alpha}(\tilde{t}))\underline{L}^{(j+1)}(\tilde{t}) - \partial_\alpha \sigma(\Delta^{(j)}\underline{\alpha}(\tilde{t}))\underline{L}^{(j)}(\tilde{t}) \right) d\tilde{t}.$$

Here, $g^{(j)}$ is the function defined by (2.13).

We will show that Φ is a contraction map on X_T^M for sufficiently small $T > 0$ and sufficiently large $M > 0$. We then obtain a unique fixed point of Φ in X_T^M and it is easily seen that the fixed point is a solution to (2.12) due to the definition of Φ.

Lemma 2.13. *There exists $T > 0$ and $M > 0$ such that the map Φ defined in Definition 2.12 satisfies $\Phi(X_T^M) \subset X_T^M$ and is a contraction map.*

Proof. For any $T > 0$, $M > c_2$ and $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) \in X_T^M$, where c_2 is the constant in Lemma 2.11, the function $\Phi(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$ obviously satisfies the condition (ii) in Lemma 2.11 since the first and last two equalities in (2.12) replaced $\Theta^{(j)}, L^{(j)}$ and $\alpha^{(j)}$ on the right hand side by $\underline{\Theta}^{(j)}, \underline{L}^{(j)}$ and $\underline{\alpha}^{(j)}$, respectively, and $\underline{\Theta}^{(j)}(\cdot,0) = \Theta_0^{(j)}, \underline{L}^{(j)}(0) = L_0^{(j)}$ and $\underline{\alpha}^{(j)}(0) = \alpha_0^{(j)}$. Therefore, it is sufficient to prove that $\Phi(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}})$ satisfies the condition (i) in Lemma 2.11 and is a contraction map for any $(\underline{\vec{\Theta}}, \underline{\vec{L}}, \underline{\vec{\alpha}}) \in X_T^M$ if $T > 0$ is sufficiently small and $M > c_2$ is sufficiently large.

To show $\Phi(X_T^M) \subset X_T^M$, we first prove that there exist $M > c_2$ and $T > 0$ such that

$$\max_{j=1,2,3} \|\Theta^{(j)}\|_{C_{x,t}^{k+\beta,\frac{k+\beta}{2}}([0,1]\times[0,T])} \le \frac{M}{3}.$$

Repeating the same argument as in (2.23), we need to prove

$$c_1 \left(\sum_{j=1}^3 \left(\|F^{(j)}\|_{C_{x,t}^{(k-2)+\beta,\frac{(k-2)+\beta}{2}}} + \|\Theta_0^{(j)}\|_{C_x^{k+\beta}} \right) + \|b_1\|_{C_t^{\frac{k+\beta}{2}}} + \|b_2\|_{C_t^{\frac{k+\beta}{2}}} + \|b_3\|_{C_t^{\frac{(k-1)+\beta}{2}}} \right) \le \frac{M}{3}. \tag{2.30}$$

Now we prove (2.30) for $k = 2$. We compute that

$$\begin{aligned}
&\left\| \left(\frac{\sigma(\Delta^{(j)}\underline{\alpha})}{(\underline{L}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) \partial_x^2 \underline{\Theta}^{(j)} \right\|_{C_{x,t}^{\beta,\frac{\beta}{2}}([0,1]\times[0,T])} \\
&\le M \|\sigma(\Delta^{(j)}\underline{\alpha})\|_{C_t^{\frac{\beta}{2}}([0,T])} \left\| \frac{1}{(\underline{L}^{(j)})^2} - \frac{1}{(L_0^{(j)})^2} \right\|_{C_t^{\frac{\beta}{2}}([0,T])} + \frac{M}{m_2^2} \|\sigma(\Delta^{(j)}\underline{\alpha}) - \sigma(\Delta^{(j)}\alpha_0)\|_{C_t^{\frac{\beta}{2}}([0,T])} \\
&\le MCm_1 \left\| \frac{1}{(\underline{L}^{(j)})^2} - \frac{1}{(L_0^{(j)})^2} \right\|_{C_t^{\frac{\beta}{2}}([0,T])} + \frac{M}{m_2^2} \|\sigma(\Delta^{(j)}\underline{\alpha}) - \sigma(\Delta^{(j)}\alpha_0)\|_{C_t^{\frac{\beta}{2}}([0,T])},
\end{aligned}$$

where $C > 0$ is a constant depending only on σ and M (if $\sigma(x) \to \infty$ as $|x| \to \infty$, C will depend on M). By

$$\left| \frac{1}{(\underline{L}^{(j)})^2} - \frac{1}{(L_0^{(j)})^2} \right| \le \frac{4}{m_2^4} |\underline{L}^{(j)} + L_0^{(j)}||\underline{L}^{(j)} - L_0^{(j)}| \le \frac{4(m_1+M)}{m_2^4} |\underline{L}^{(j)} - L_0^{(j)}|$$

and

$$\left| \frac{1}{(\underline{L}^{(j)}(s))^2} - \frac{1}{(\underline{L}^{(j)}(t))^2} \right| \le \frac{8}{m_2^4} |\underline{L}^{(j)}(s) + \underline{L}^{(j)}(t)||\underline{L}^{(j)}(s) - \underline{L}^{(j)}(t)| \le \frac{16M}{m_2^4} |\underline{L}^{(j)}(s) - \underline{L}^{(j)}(t)|,$$

we have

$$\begin{aligned}
&\left\| \frac{1}{(\underline{L}^{(j)})^2} - \frac{1}{(L_0^{(j)})^2} \right\|_{C_t^{\frac{\beta}{2}}([0,T])} \\
&\le \frac{20(m_1+M)}{m_2^4} \left\{ \|\underline{L}^{(j)} - L_0^{(j)}\|_{C^0([0,T])} + \sup_{t,s\in[0,T],t\neq s} \frac{|\underline{L}^{(j)}(s) - \underline{L}^{(j)}(t)|}{|s-t|^{\frac{\beta}{2}}} \right\} \\
&\le \frac{20(m_1+M)}{m_2^4} \left\{ \|\underline{L}^{(j)}\|_{C^{\frac{\beta'}{2}}([0,T])} T^{\frac{\beta'}{2}} + \sup_{t,s\in[0,T],t\neq s} \frac{|\underline{L}^{(j)}(s) - \underline{L}^{(j)}(t)|}{|s-t|^{\frac{\beta'}{2}}} |s-t|^{\frac{\beta'-\beta}{2}} \right\} \\
&\le \frac{20(m_1+M)}{m_2^4} \|\underline{L}^{(j)}\|_{C^{\frac{\beta'}{2}}([0,T])} \left\{ T^{\frac{\beta'}{2}} + T^{\frac{\beta'-\beta}{2}} \right\}.
\end{aligned} \tag{2.31}$$

Similarly, there exists $C = C(m_1, m_2, M, \sigma) > 0$ such that

$$\|\sigma(\Delta^{(j)}\underline{\alpha}) - \sigma(\Delta^{(j)}\alpha_0)\|_{C_t^{\frac{\beta}{2}}([0,T])} \le C \left\{ T^{\frac{\beta'}{2}} + T^{\frac{\beta'-\beta}{2}} \right\}.$$

Therefore

$$\left\| \left(\frac{\sigma(\Delta^{(j)}\underline{\alpha})}{(\underline{L}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) \partial_x^2 \underline{\Theta}^{(j)} \right\|_{C_{x,t}^{\beta,\frac{\beta}{2}}([0,1]\times[0,T])} \le C \left\{ T^{\frac{\beta'}{2}} + T^{\frac{\beta'-\beta}{2}} \right\},$$

where $C > 0$ depends only on m_1, m_2, M, and σ. By repeating the same argument for $f^{(j)}$ and b_i, there exist $C = C(m_1, m_2, M, \sigma) > 0$, $C' = C'(m_1, m_2, \sigma) > 0$, and $\gamma = \gamma(\beta, \beta') > 0$ such that

$$\sum_{j=1}^{3} \|F^{(j)}\|_{C_{x,t}^{\beta,\frac{\beta}{2}}} + \|b_1\|_{C_t^{\frac{\beta}{2}}} + \|b_2\|_{C_t^{\frac{\beta}{2}}} + \|b_3\|_{C_t^{\frac{\beta}{2}}} \le CT^{\gamma} + C'.$$

Note that C' is independent of M. For the readers' convenience, we explain why C' is necessary and is independent of M. For $t \in (0,T]$, in order to estimate the C^β norm of $\partial_x\Theta^{(j)}(x,t)$ with respect to x, we have

$$\begin{aligned}
&\frac{|\partial_x\Theta^{(j)}(x,t)-\partial_x\Theta^{(j)}(y,t)|}{|x-y|^\beta}\\
&\leq \frac{|\partial_x\Theta^{(j)}(x,0)-\partial_x\Theta^{(j)}(y,0)|}{|x-y|^\beta}+\frac{|\partial_x\Theta^{(j)}(x,t)-\partial_x\Theta^{(j)}(x,0)|}{|x-y|^\beta}+\frac{|\partial_x\Theta^{(j)}(y,t)-\partial_x\Theta^{(j)}(y,0)|}{|x-y|^\beta}\\
&\leq m_1+\frac{|\partial_x\Theta^{(j)}(x,t)-\partial_x\Theta^{(j)}(x,0)|}{|x-y|^\beta}+\frac{|\partial_x\Theta^{(j)}(y,t)-\partial_x\Theta^{(j)}(y,0)|}{|x-y|^\beta}
\end{aligned} \tag{2.32}$$

for any $0 \leq x < y \leq 1$. In the case of $y-x > \sqrt{t}$, we have

$$\frac{|\partial_x\Theta^{(j)}(x,t)-\partial_x\Theta^{(j)}(x,0)|}{|x-y|^\beta} \leq Mt^{\frac{1+\beta}{2}-\frac{\beta}{2}} = Mt^{\frac{1}{2}},$$

where we used $\sup_{0\leq t<s\leq T, x\in[0,1]} \frac{|\partial_x\Theta^{(j)}(x,s)-\partial_x\Theta^{(j)}(x,t)|}{|s-t|^{\frac{1+\beta}{2}}} \leq M$. On the other hand, if $y-x \leq \sqrt{t}$,

$$\frac{|\partial_x\Theta^{(j)}(x,t)-\partial_x\Theta^{(j)}(y,t)|}{|x-y|^\beta} \leq M|x-y|^{1-\beta} \leq Mt^{\frac{1-\beta}{2}},$$

where we used $\|\partial_{xx}\Theta^{(j)}\|_{C^0_{x,t}} \leq M$. Thus we obtain

$$\sup_{x,y\in[0,1],x\neq y} \frac{|\partial_x\Theta^{(j)}(x,t)-\partial_x\Theta^{(j)}(y,t)|}{|x-y|^\beta} \leq M(t^{\frac{1}{2}}+t^{\frac{1-\beta}{2}})+m_1.$$

The second term of the right hand side corresponds to C' above. For nonlinear terms, similar estimates hold. Note that for sufficiently small $T>0$ there exists $C>0$ depending only on M such that $C < 1-c^{(1)}c^{(2)}c^{(3)} \leq 1$. Since $\|\phi(1,\cdot)\|_{C_t^{\frac{\beta}{2}}} \leq \|\phi\|_{C_{x,t}^{\beta,\frac{\beta}{2}}}$ for $\phi=\phi(x,t)$, the estimate $\|b_i\|_{C_t^{\frac{\beta}{2}}}$ $(i=1,2,3)$ can be done as above.

Therefore, we obtain (2.30) with $k=2$ for sufficiently small $T>0$ and sufficiently large $M>0$. Repeating the same argument above, we may obtain (2.30) and

$$\max_{j=1,2,3}\|\Theta^{(j)}\|_{C^{k+\beta}_{x,t}}+\max_{j=1,2,3}\|L^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}}+\max_{j=1,2,3}\|\alpha^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}} \leq M$$

for the case of $k\geq 3$. In addition, there exists $C>0$ depending only on M such that

$$\sup_{\tilde{t}\in[0,T)}\left| g^{(j)}(\underline{\vec{\Theta}},\underline{\vec{\alpha}})(\tilde{t}) - \frac{\sigma(\Delta^{(j)}\underline{\alpha}(\tilde{t}))}{\underline{L}^{(j)}(\tilde{t})}\int_0^1 (\partial_x\underline{\Theta}^{(j)}(x,\tilde{t}))^2\,dx\right| \leq C.$$

Therefore $\inf_{t\in[0,T]} L^{(j)}(t) \geq \frac{m_2}{2}$ for sufficiently small $T>0$. Hence we obtain $\Phi(X_T^M)\subset X_T^M$.

Finally we show that Φ is a contraction mapping. Assume that $(\underline{\vec{\Theta}_i},\underline{\vec{L}_i},\underline{\vec{\alpha}_i}) \in X_T^M$ and $(\vec{\Theta}_i,\vec{L}_i,\vec{\alpha}_i) = \Phi(\underline{\vec{\Theta}_i},\underline{\vec{L}_i},\underline{\vec{\alpha}_i})$ for $i=1,2$. Set $(\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}}) := (\underline{\vec{\Theta}_1},\underline{\vec{L}_1},\underline{\vec{\alpha}_1})-(\underline{\vec{\Theta}_2},\underline{\vec{L}_2},\underline{\vec{\alpha}_2})$. Then $(\vec{\Theta},\vec{L},\vec{\alpha}) := (\vec{\Theta}_1,\vec{L}_1,\vec{\alpha}_1)-(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_2)$ is a solution of (2.14) with $F^{(j)}=F^{(j)}(\vec{\Theta}_1,\vec{L}_1,\vec{\alpha}_1)-F^{(j)}(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_2)$ and $b_j=b_j(\vec{\Theta}_1,\vec{L}_1,\vec{\alpha}_1)-b_j(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_2)$ for $j=1,2,3$. We prove that for sufficiently small $T>0$ it holds that

$$\max_{j=1,2,3}\|\Theta^{(j)}\|_{C^{k+\beta}_{x,t}}+\max_{j=1,2,3}\|L^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}}+\max_{j=1,2,3}\|\alpha^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}} \leq \frac{M'}{2}, \tag{2.33}$$

where

$$M' = \max_{j=1,2,3} \|\underline{\Theta}^{(j)}\|_{C^{k+\beta}_{x,t}} + \max_{j=1,2,3} \|\underline{L}^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}} + \max_{j=1,2,3} \|\underline{\alpha}^{(j)}\|_{C_t^{\frac{k+\beta'}{2}}}.$$

By (2.23), we have

$$\sum_{j=1}^{3} \|\Theta^{(j)}\|_{C^{k+\beta,\frac{k+\beta}{2}}_{x,t}} \le c_1 \left(\sum_{j=1}^{3} \|F^{(j)}\|_{C^{(k-2)+\beta,\frac{(k-2)+\beta}{2}}_{x,t}} + \|b_1\|_{C_t^{\frac{k+\beta}{2}}} + \|b_2\|_{C_t^{\frac{k+\beta}{2}}} + \|b_3\|_{C_t^{\frac{(k-1)+\beta}{2}}} \right),$$

where we used $(\vec{\Theta},\vec{L},\vec{\alpha})|_{t=0} = \vec{0}$. By $(\underline{\vec{\Theta}},\underline{\vec{L}},\underline{\vec{\alpha}})|_{t=0} = \vec{0}$, there exists $\gamma > 0$ such that

$$\sum_{j=1}^{3} \|F^{(j)}\|_{C^{(k-2)+\beta,\frac{(k-2)+\beta}{2}}_{x,t}} + \|b_1\|_{C_t^{\frac{k+\beta}{2}}} + \|b_2\|_{C_t^{\frac{k+\beta}{2}}} + \|b_3\|_{C_t^{\frac{(k-1)+\beta}{2}}} \le c_3 M' T^{\gamma}, \tag{2.34}$$

where $c_3 = c_3(m_1, m_2, M, \sigma) > 0$. For example, in the case of $k = 2$, we have

$$\begin{aligned}
\|F^{(j)}\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} &= \|F^{(j)}(\vec{\Theta}_1,\vec{L}_1,\vec{\alpha}_1) - F^{(j)}(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_2)\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} \\
&\le \|F^{(j)}(\vec{\Theta}_1,\vec{L}_1,\vec{\alpha}_1) - F^{(j)}(\vec{\Theta}_2,\vec{L}_1,\vec{\alpha}_1)\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} + \|F^{(j)}(\vec{\Theta}_2,\vec{L}_1,\vec{\alpha}_1) - F^{(j)}(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_1)\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} \\
&\quad + \|F^{(j)}(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_1) - F^{(j)}(\vec{\Theta}_2,\vec{L}_2,\vec{\alpha}_2)\|_{C^{\beta,\frac{\beta}{2}}_{x,t}}.
\end{aligned}$$

Now we calculate

$$\mathcal{F} := \left(\frac{\sigma(\Delta^{(j)}\underline{\alpha_1})}{(\underline{L_1}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) \partial_x^2 \underline{\Theta_1}^{(j)} - \left(\frac{\sigma(\Delta^{(j)}\underline{\alpha_1})}{(\underline{L_1}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) \partial_x^2 \underline{\Theta_2}^{(j)}$$

contained in $F^{(j)}(\vec{\Theta}_1,\vec{L}_1,\vec{\alpha}_1) - F^{(j)}(\vec{\Theta}_2,\vec{L}_1,\vec{\alpha}_1)$. By an argument similar to that in (2.31), we compute

$$\begin{aligned}
\|\mathcal{F}\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} &= \left\| \left(\frac{\sigma(\Delta^{(j)}\underline{\alpha_1})}{(\underline{L_1}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right) (\partial_x^2 \underline{\Theta_1}^{(j)} - \partial_x^2 \underline{\Theta_2}^{(j)}) \right\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} \\
&\le \left\| \frac{\sigma(\Delta^{(j)}\underline{\alpha_1})}{(\underline{L_1}^{(j)})^2} - \frac{\sigma(\Delta^{(j)}\alpha_0)}{(L_0^{(j)})^2} \right\|_{C_t^{\frac{\beta}{2}}} \left\| \partial_x^2 \underline{\Theta}^{(j)} \right\|_{C^{\beta,\frac{\beta}{2}}_{x,t}} \le CM' \left\{ T^{\frac{\beta'}{2}} + T^{\frac{\beta'-\beta}{2}} \right\}.
\end{aligned}$$

Similar computation leads (2.34). Then we choose $T > 0$ such that $c_3 T^{\gamma} \le \frac{M'}{10c_1}$. Therefore we have $\max_{j=1,2,3} \|\Theta^{(j)}\|_{C^{k+\beta,\frac{k+\beta}{2}}_{x,t}} \le \frac{M'}{10}$. Similarly we can obtain (2.33). □

The following existence theory for the system (2.12) immediately follows since we constructed the contraction map associated to the system.

Proposition 2.14. *Let $k \in \mathbb{N}$ with $k \ge 2$ and $\beta \in (0,1)$. Assume $\vec{\Theta}_0 \in (C_x^{k+\beta}([0,1]))^3$, $\vec{L}_0 \in \{(0,\infty)\}^3$ and $\vec{\alpha}_0 \in \mathbb{R}^3$ satisfy the compatibility condition of order k for (2.12) and $\Theta_0^{(j+1)}(1) - \Theta_0^{(j)}(1) \in (0,\pi)$ for $j = 1,2,3$. Then, there exist $T > 0$ such that the system (2.12) have a unique solution $(\vec{\Theta},\vec{L},\vec{\alpha}) \in (C_{x,t}^{k+\beta,\frac{k+\beta}{2}}([0,1]\times[0,T]))^3 \times (C_t^{\frac{k+\beta}{2}}([0,T]))^3 \times (C_t^{\frac{k+\beta}{2}}([0,T]))^3$ starting from $(\vec{\Theta}_0,\vec{L}_0,\vec{\alpha}_0)$.*

The system (2.12) was derived from the geometric flow (1.3)–(1.7) and, conversely, the geometric flow can be constructed from the solution to the system. We thus obtain the existence theorem for the geometric flow as in Theorem 1.1.

Proof of Theorem 1.1. Let $L_0^{(j)}$ be the length of $\Gamma_0^{(j)}$ and parametrize $\Gamma_0^{(j)}$ by $x = s/L_0^{(j)}$, where s is the arc-length of $\Gamma_0^{(j)}$. Define the angle function $\Theta_0^{(j)}$ of $\Gamma_0^{(j)}$ as in (2.1). We then easily see that $\Theta_0^{(j)}$ is of class $C_x^{(k-1)+\beta}([0,1])$, $\Theta_0^{(j+1)}(1) - \Theta_0^{(j)}(1)$ is positive and less than π, and $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ satisfies the compatibility condition of order $k-1$ for (2.12). Therefore, for any $\beta' \in (\beta, 1)$ and some $T > 0$, the unique solution $(\vec{\Theta}, \vec{L}, \vec{\alpha}) \in (C_{x,t}^{(k-1)+\beta, \frac{(k-1)+\beta}{2}}([0,1]\times[0,T]))^3 \times (C_t^{\frac{(k-1)+\beta'}{2}}([0,T]))^3 \times (C_t^{\frac{(k-1)+\beta'}{2}}([0,T]))^3$ to (2.12) starting from $(\vec{\Theta}_0, \vec{L}_0, \vec{\alpha}_0)$ can be obtained due to Proposition 2.14. Letting

$$\xi^{(j)}(x,t) := \left(\int_0^x L^{(j)}(t)\cos\Theta^{(j)}(\tilde{x},t)\, d\tilde{x}, \int_0^x L^{(j)}(t)\sin\Theta^{(j)}(\tilde{x},t)\, d\tilde{x}\right) + P^{(j)}$$

for $x \in [0,1]$ and $t \geq 0$, where $P^{(j)}$ is the fixed point in (1.7), we can also easily see that the curve $\Gamma_t^{(j)} := \xi^{(j)}([0,1],t)$ starts from $\Gamma_0^{(j)}$ for $j = 1,2,3$. We now prove that $\{\Gamma_t^{(j)}\}$ and $\vec{\alpha}$ satisfy the geometric flow equations (1.3)–(1.7) until the time T. Since $f^{(j)}$ defined by (2.13) can be re-written as, by means of (2.12),

$$f^{(j)} = \left(\frac{\sigma(\Delta^{(j)}\alpha)}{(L^{(j)})^2}\int_0^x (\partial_x\Theta^{(j)}(\tilde{x},t))^2\, d\tilde{x} + \frac{x}{L^{(j)}}\partial_t L^{(j)}\right)\partial_x\Theta^{(j)},$$

we have by (2.12) and integration by parts

$$\begin{aligned}
&\partial_t \int_0^x L^{(j)}(t)\cos\Theta^{(j)}(\tilde{x},t)\, d\tilde{x} \\
&= \int_0^x \partial_t L^{(j)}\cos\Theta^{(j)} - \left\{\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\partial_x^2\Theta + \left(\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\int_0^{\tilde{x}}(\partial_x\Theta^{(j)})^2\, d\hat{x} + \tilde{x}\partial_t L^{(j)}\right)\partial_x\Theta^{(j)}\right\}\sin\Theta^{(j)}\, d\tilde{x} \\
&= \int_0^x \partial_t L^{(j)}\partial_x(\tilde{x}\cos\Theta^{(j)}) + \frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\partial_x\left(\int_0^{\tilde{x}}(\partial_x\Theta^{(j)})^2\, d\hat{x}\cos\Theta^{(j)}\right)\, d\tilde{x} - \frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\partial_x\Theta^{(j)}\sin\Theta^{(j)} \\
&= -\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\partial_x\Theta^{(j)}\sin\Theta^{(j)} + \left(\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\int_0^x(\partial_x\Theta^{(j)})^2\, d\hat{x} + x\partial_t L^{(j)}\right)\cos\Theta^{(j)}.
\end{aligned}$$

Since $\partial_x\Theta^{(j)}/L^{(j)}$ is the curvature $\kappa_t^{(j)}$ of $\Gamma_t^{(j)}$ and (2.1) holds for $\Gamma_t^{(j)}$, we can obtain similarly

$$\partial_t\xi^{(j)} = \sigma(\Delta^{(j)}\alpha)\kappa_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)},$$

where

$$\lambda_t^{(j)} = \left(\frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}}\int_0^x(\partial_x\Theta^{(j)})^2\, d\hat{x} + x\partial_t L^{(j)}\right) \tag{2.35}$$

and $\tau_t^{(j)}, \nu_t^{(j)}$ are the unit tangent and normal vector of $\Gamma_t^{(j)}$ respectively. Therefore, (1.3) is satisfied. The equations (1.4), (1.7) hold obviously. Note that the third and forth equations in (2.12) are equivalent to

$$\sum_{j=1}^3 \sigma(\Delta^{(j)}\alpha)\tau_t^{(j)} = \sum_{j=1}^3 \sigma(\Delta^{(j)}\alpha)(\cos\Theta, \sin\Theta) = 0 \quad \text{at} \ \ x = 1, \tag{2.36}$$

and thus (1.6) is satisfied if (1.5) holds. We now thus prove (1.5). By means of the form (2.35) of the tangent velocity and the differential equation of $L^{(j)}$ in (2.12), we can see that $\lambda_t^{(j)}$ satisfies (2.7).

Hereafter, we use $V_t^{(j)}$ as the normal velocity of $\Gamma_t^{(j)}$ and thus

$$V_t^{(j)} = \frac{\sigma(\Delta^{(j)}\alpha)}{L^{(j)}(t)}\partial_x\Theta^{(j)} = \sigma(\Delta^{(j)}\alpha)\partial_s\Theta^{(j)} = \sigma(\Delta^{(j)}\alpha)\kappa_t^{(j)},$$

and also let $c^{(j)}(t) = \cos(\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t))$ and $s^{(j)}(t) = \sin(\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t))$ for simplicity. We then have by the fifth equality in (2.12) and the sum and difference formulas of trigonometric functions

$$\sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha)V_t^{(j)} = 0, \tag{2.37}$$

$$\begin{aligned} c^{(k)}(t) =& \cos\Big((\Theta^{(k+1)}(1,t) - \Theta^{(k+2)}(1,t)) + (\Theta^{(k+2)}(1,t) - \Theta^{(k)}(1,t))\Big) \\ &= c^{(k+1)}c^{(k+2)} - s^{(k+1)}s^{(k+2)} \quad \text{for } k \in \{1,2,3\}. \end{aligned} \tag{2.38}$$

Furthermore, taking inner product of (2.36) and $\nu_t^{(k)}$ at $x = 1$, we have

$$\sigma(\Delta^{(k+1)}\alpha)s^{(k)} = \sigma(\Delta^{(k+2)}\alpha)s^{(k+2)} \quad \text{for } k \in \{1,2,3\}. \tag{2.39}$$

In order to prove (1.5), it is sufficient to prove, for $j \in \{1,2,3\}$ and $t \geq 0$, $\partial_t\xi^{(j)} - \partial_t\xi^{(j+1)} = 0$ at $x = 1$, which is equivalent to

$$(V_t^{(j)}\nu_t^{(j)} + \lambda_t^{(j)}\tau_t^{(j)}) - (V_t^{(j+1)}\nu_t^{(j+1)} + \lambda_t^{(j+1)}\tau_t^{(j+1)}) = 0 \quad \text{at } x = 1. \tag{2.40}$$

Our strategy is to prove that the each inner product of (2.40) and $\tau_t^{(j)}/\nu_t^{(j)}$ is zero for any $j \in \{1,2,3\}$. The inner product of (2.40) and $\tau_t^{(j)}$ is equivalent to $\lambda_t^{(j)} = -s^{(j)}V_t^{(j+1)} + c^{(j)}\lambda_t^{(j+1)}$ and this identity obviously holds due to (2.7). We next take inner product of the left hand side of (2.40) and $\nu_t^{(j)}$ at $x = 1$ to obtain

$$\begin{aligned} \langle\partial_t\xi^{(j)} - \partial_t\xi^{(j+1)}, \nu_t^{(j)}\rangle &= V_t^{(j)} - c^{(j)}V_t^{(j+1)} - s^{(j)}\lambda_t^{(j+1)} \\ &= \frac{1}{1 - c^{(j)}c^{(j+1)}c^{(j+2)}}\Big((1 - c^{(j)}c^{(j+1)}c^{(j+2)} + s^{(j)}c^{(j+1)}s^{(j+2)})V^{(j)} \\ &\qquad + (-c^{(j)}(1 - c^{(j)}c^{(j+1)}c^{(j+2)}) + (s^{(j)})^2c^{(j+1)}c^{(j+2)})V^{(j+1)} + s^{(j)}s^{(j+1)}V^{(j+2)}\Big) \end{aligned}$$

at $x = 1$. Here, the following identity, which follows from (2.7), have been used:

$$\lambda_t^{(j+1)} = \frac{-1}{1 - c^{(j)}c^{(j+1)}c^{(j+2)}}(c^{(j+1)}s^{(j+2)}V_t^{(j)} + s^{(j)}c^{(j+1)}c^{(j+2)}V_t^{(j+1)} + s^{(j+1)}V_t^{(j+2)}) \quad \text{at } x = 1.$$

Since (2.38) implies

$$\begin{aligned} &1 - c^{(j)}c^{(j+1)}c^{(j+2)} + s^{(j)}c^{(j+1)}s^{(j+2)} = 1 - (c^{(j+1)})^2 = (s^{(j+1)})^2, \\ &- c^{(j)}(1 - c^{(j)}c^{(j+1)}c^{(j+2)}) + (s^{(j)})^2c^{(j+1)}c^{(j+2)} = c^{(j+1)}c^{(j+2)} - c^{(j)} = s^{(j+1)}s^{(j+2)}, \end{aligned}$$

by means of (2.37) and (2.39), we have

$$\begin{aligned} \langle\partial_t\xi^{(j)} - \partial_t\xi^{(j+1)}, \nu_t^{(j)}\rangle &= \frac{1}{1 - c^{(j)}c^{(j+1)}c^{(j+2)}}((s^{(j+1)})^2V_t^{(j)} + s^{(j+1)}s^{(j+2)}V_t^{(j+1)} + s^{(j)}s^{(j+1)}V_t^{(j+2)}) \\ &= \frac{s^{(j)}s^{(j+1)}}{\sigma(\Delta^{(j+2)}\alpha)(1 - c^{(j)}c^{(j+1)}c^{(j+2)})}\sum_{k=1}^{3}\sigma(\Delta^{(k)}\alpha)V_t^{(k)} = 0 \quad \text{at } x = 1. \end{aligned}$$

Therefore, $\partial_t \xi^{(j)} - \partial_t \xi^{(j+1)}$ is zero at $x = 1$ in each direction $\nu_t^{(j)}$ and $\tau_t^{(j)}$, and thus (2.40) holds since $\nu_t^{(j)}$ and $\tau_t^{(j)}$ are linearly independent. Since the initial triod satisfies (1.5), the identity (2.40) show that (1.5) holds for any positive time t.

Next, to prove the uniqueness of the geometric flow, assume that a family of $\tilde{\Gamma}_t^{(j)}$ and $\tilde{\alpha}^{(j)}$, starting from $\Gamma_0^{(j)}$ and $\alpha_0^{(j)}$ respectively, is another solution to (1.3)–(1.7). We then see that the family of the angle function $\tilde{\Theta}^{(j)}$ of $\tilde{\Gamma}_t^{(j)}$, the length $\tilde{L}^{(j)}$ of $\tilde{\Gamma}_t^{(j)}$ and the orientation parameter $\tilde{\alpha}^{(j)}$ satisfy (2.12) with the variables $\tilde{x} = \tilde{s}/\tilde{L}^{(j)}$ on $\tilde{\Gamma}_t^{(j)}$, where $\tilde{s}$ is the arc-length of $\tilde{\Gamma}_t^{(j)}$. Due to the uniqueness result for (2.12), we obtain $\tilde{\Theta}^{(j)} \equiv \Theta^{(j)}$, $\tilde{L}^{(j)} \equiv L^{(j)}$ and $\tilde{\alpha}^{(j)} \equiv \alpha^{(j)}$, which implies coincidence of two geometric flows.

Finally, we prove that the geometric flow is smooth if $\Gamma_0^{(j)}$ is smooth and the pair of $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_0^{(j)}\}_{j\in\{1,2,3\}}$ satisfies the compatibility condition of any order for the geometric flow (1.3)–(1.7). Proposition 2.14 shows that, for any integer $k \geq 2$, there exists $T_k > 0$ such that the angle function $\Theta^{(j)}$ of $\Gamma_t^{(j)}$ satisfies

$$\Theta^{(j)} \in C_{x,t}^{k+\beta,\frac{k+\beta}{2}}([0,1] \times [0,T_k]) \quad \text{for} \quad j \in \{1,2,3\}, \quad k \in \mathbb{N} \quad \text{with} \quad k \geq 2. \tag{2.41}$$

Note that, by means of the differential equations of $L^{(j)}$ and $\alpha^{(j)}$ in (2.12), we also see that

$$L^{(j)} \in C_t^{\frac{k+1/2+\beta}{2}}([0,T_k]), \quad \alpha^{(j)} \in C_t^{\frac{k+3/2+\beta}{2}}([0,T_k]), \quad \text{for} \quad j \in \{1,2,3\}, \quad k \in \mathbb{N} \quad \text{with} \quad k \geq 2. \tag{2.42}$$

Therefore, it is sufficient to prove that there exists $T' > 0$ such that $T_k \geq T'$ for any k. Letting $v^{(j)} := \partial_t \Theta^{(j)}$, it follows from the regularity results (2.41), (2.42) and the differential equations of $\Theta^{(j)}$ in (2.12) that v satisfies

$$\begin{cases} \partial_t v^{(j)} = \frac{\sigma(\Delta^{(j)}\alpha)}{(L^{(j)})^2}\partial_x^2 v + \tilde{F}^{(j)}, & (x,t) \in (0,1) \times (0,T_k], \quad j = 1,2,3, \\ \partial_x v^{(j)}(0,t) = 0, & t \in (0,T_k], \quad j = 1,2,3, \\ \sum_{j=1}^3 \left(\sigma(\Delta^{(j)}\alpha(t)) \sin\Theta^{(j)}(1,t)\right) v^{(j)}(1,t) = \tilde{b}_1(t), & t \in (0,T_k], \\ \sum_{j=1}^3 \left(\sigma(\Delta^{(j)}\alpha(t)) \cos\Theta^{(j)}(1,t)\right) v^{(j)}(1,t) = \tilde{b}_2(t), & t \in (0,T_k], \\ \sum_{j=1}^3 \frac{(\sigma(\Delta^{(j)}\alpha(t)))^2}{L^{(j)}(t)} \partial_x v^{(j)}(1,t) = \tilde{b}_3(t), & t \in (0,T_k], \end{cases} \tag{2.43}$$

where $\tilde{F}^{(j)}, \tilde{b}_j$ are some functions of class

$$\tilde{F}^{(j)} \in C_{x,t}^{k-3+\beta,\frac{k-3+\beta}{2}}([0,1] \times [0,T_k]) \quad \text{for} \quad j \in \{1,2,3\},$$
$$\tilde{b}_j \in C_t^{\frac{k+\beta}{2}}([0,T_k]) \quad \text{for} \quad j \in \{1,2\}, \quad \tilde{b}_3 \in C_t^{\frac{k-1+\beta}{2}}([0,T_k]),$$

for any $k \geq 4$. While some coefficients in (2.43) depend on t, we can prove that, by a similar argument as in Section 2.2, the system (2.43) satisfies the parabolicity and the complementing condition whenever $\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t) \in (0,\pi)$. The condition $\Theta^{(j+1)}(1,t) - \Theta^{(j)}(1,t) \in (0,\pi)$ holds on some time interval $[0,T_0]$ since $\Theta^{(j)}$ is continuous and the condition is satisfied at initial time. Furthermore, due to the assumptions on the initial datum for the geometric flow, $v(\cdot,0)$ satisfies the compatibility condition of any order for (2.43). It is thus possible to apply [26, Theorem 10.1 in Chapter VII], as in Proposition 2.10, to conclude that $v^{(j)} \in C_{x,t}^{k-1+\beta,\frac{k-1+\beta}{2}}([0,1] \times [0,T_k])$ for $j \in \{1,2,3\}$ and $k \geq 4$. Since $\partial_x^2 \Theta^{(j)}$ can be represented by a summation of some terms containing $v^{(j)}$ and $\partial_x \Theta^{(j)}$ as in the first differential equation in (2.12), $\partial_x^2 \Theta^{(j)}$ has same regularity with v. Therefore, we obtain $\Theta^{(j)} \in C^{k+1+\beta,\frac{k+1+\beta}{2}}_{x,t}([0,1]\times[0,T_k])$ for $j \in \{1,2,3\}$ and $k \geq 4$ (see also [31, Theorem 2.2] for the regularity result of $\Theta^{(j)}$ from it of $v^{(j)}$ and $\partial_x^2\Theta^{(j)}$). It shows $T_{k+1} \geq \min\{T_k, T_0\}$ and further $T_k \geq \min\{T_4, T_0\}$ for any $k \geq 4$. □

3. Equilibrium

In this section, we study the equilibrium state for (1.3)–(1.7) to continue to prove the local exponential stability of the steady state. In particular, we show that the steady state is unique up to constant difference for $(\alpha^{(1)},\alpha^{(2)},\alpha^{(3)})$ under the assumptions (A1)–(A3).

Proposition 3.1. *Let $\gamma,\sigma,P^{(j)}(j=1,2,3)$ satisfy the assumptions (A1)–(A3). Let also $(\Gamma^{(j)}_\infty,\alpha^{(j)}_\infty)$ be a stationary solution to* (1.3)–(1.7)*. Then, $\cup_{j=1}^3\Gamma^{(j)}_\infty$ is the unique Steiner triod connectiong the three boundary points $P^{(j)}$ and $\alpha^{(1)}_\infty=\alpha^{(2)}_\infty=\alpha^{(3)}_\infty$.*

The latter property for $\alpha^{(j)}_\infty$ can be obtain by a quite similar argument in [11, Proposition 5.1]. We however give the proof of the property for self-contained arguments.

Proof. Let $L^{(j)}_\infty$ be the length of $\Gamma^{(j)}_\infty$. We first prove that $\alpha^{(1)}_\infty=\alpha^{(2)}_\infty=\alpha^{(3)}_\infty$. Since $(\Gamma^{(j)}_\infty,\alpha^{(j)}_\infty)$ is a stationary solution, we have

$$\partial_\alpha\sigma(\Delta^{(1)}\alpha_\infty)L^{(1)}_\infty=\partial_\alpha\sigma(\Delta^{(2)}\alpha_\infty)L^{(2)}_\infty=\partial_\alpha\sigma(\Delta^{(3)}\alpha_\infty)L^{(3)}_\infty, \tag{3.1}$$

which is derived from (1.4). We also note that the assumption (A3) implies

$$\partial_\alpha\sigma(\alpha)\alpha\geq 0 \quad \text{for} \quad \alpha\in\mathbb{R}. \tag{3.2}$$

Multiplying $\Delta^{(2)}\alpha_\infty$ and $\Delta^{(3)}\alpha_\infty$ to (3.1) and using convexity of σ as in (3.2), we find

$$\partial_\alpha\sigma(\Delta^{(1)}\alpha_\infty)\Delta^{(2)}\alpha_\infty L^{(1)}_\infty\geq 0, \quad \text{and} \quad \partial_\alpha\sigma(\Delta^{(1)}\alpha_\infty)\Delta^{(3)}\alpha_\infty L^{(1)}_\infty\geq 0, \tag{3.3}$$

which imply $\partial_\alpha\sigma(\Delta^{(1)}\alpha_\infty)\Delta^{(1)}\alpha_\infty\leq 0$ by taking the summation and dividing by $L^{(1)}_\infty$. Combining it and (3.2), we obtain $\partial_\alpha\sigma(\Delta^{(1)}\alpha_\infty)\Delta^{(1)}\alpha_\infty=0$, which implies $\Delta^{(1)}\alpha_\infty=0$ due to the assumption (A3). As a similar argument, we obtain $\Delta^{(2)}\alpha_\infty=\Delta^{(3)}\alpha_\infty=0$ and thus $\alpha^{(1)}_\infty=\alpha^{(2)}_\infty=\alpha^{(3)}_\infty$ holds.

We next prove that the network $\cup_{j=1}^3\Gamma^{(j)}_\infty$ is the unique Steiner triod connecting the three boundary points $P^{(j)}$. Since $(\Gamma^{(j)}_\infty,\alpha^{(j)}_\infty)$ is a stationary solution, the curvature of $\Gamma^{(j)}_\infty$ is 0 for any $j\in\{1,2,3\}$ in view of (1.3). Therefore, $\Gamma^{(j)}_\infty$ is a line segment for $j\in\{1,2,3\}$ and the line segments generate a network connecting $P^{(1)},P^{(2)},P^{(3)}$ and a junction point p_∞. Furthermore, due to $\Delta^{(1)}\alpha_\infty=\Delta^{(2)}\alpha_\infty=\Delta^{(3)}\alpha_\infty=0$, the each pair of line segments in $\{\Gamma^{(j)}_\infty\}_{j\in\{1,2,3\}}$ generates 120 degree contact angle at the junction point p_∞ in view of (1.6). We thus conclude that the network is the unique Steiner triod connecting the three boundary points $P^{(j)}$ due to the assumption (A2). □

Remark 3.2. If the assumption (A3) is dropped in Proposition 3.1, we possibly obtain more than two networks of equilibrium. One of the most simple case losing the uniqueness of the network is that $\partial_\alpha\sigma$ has other zero points a,b with $2a+b=0$ besides $\alpha=0$. In this case, we can choose the orientation parameters to satisfy $\Delta^{(1)}\alpha_\infty=\Delta^{(2)}\alpha_\infty=a$ and $\Delta^{(3)}\alpha_\infty=b$. Furthremore, if we take an appropriate function σ and the fixed points $P^{(j)}$, there is a network consists of line segments $\Gamma^{(j)}_\infty$ between a junction point $\vec{a}_\infty$ and $P^{(j)}$ satisfying

$$\sum_{j=1}^3\sigma(\Delta^{(j)}\alpha_\infty)\tau^{(j)}_\infty=0.$$

Since this triod is obviously different from the Steiner triod if $\sigma(a) \neq \sigma(b)$, we can see the non-uniqueness of the geometric form of the equilibrium. We note additionally that an equilibrium satisfying $\partial_\alpha \sigma(\Delta^{(j)}\alpha_\infty) \neq 0$ for $j \in \{1, 2, 3\}$ also can be constructed if we choose σ and $P^{(j)}$ properly.

4. Geometric properties and exponential decay of misorientations

In this section, we assume that a smooth geometric flow governed by (1.3)–(1.7) exists until a time $T > 0$. Let $\xi^{(j)}$ be the parametrization of $\Gamma_t^{(j)}$ defined as in Section 2.1 without the restriction (2.9). According to the discussions in Section 2, the uniformly boundedness of $L^{(j)}$ from below and above is required to ensure the uniform parabolicity of the problem, and we thus first study the boundedness. The boundedness of $L^{(j)}$ from above can be obtained from the energy dissipation.

Lemma 4.1. *Assume (A1). Let*

$$E(t) := \sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} \sigma(\Delta^{(j)}\alpha(t))\, ds.$$

Then,

$$\frac{d}{dt}E(t) = -\sum_{j=1}^{3} \left(\int_{\Gamma_t^{(j)}} (V_t^{(j)})^2\, ds + \frac{(\partial_t \alpha^{(j)}(t))^2}{\gamma} \right). \tag{4.1}$$

Proof. Applying (2.3), we have by $V_t^{(j)} = \sigma(\Delta^{(j)}\alpha(t))\partial_s \Theta^{(j)}$

$$\begin{aligned} \frac{d}{dt}E(t) &= \sum_{j=1}^{3} \frac{d}{dt} \int_0^1 \sigma(\Delta^{(j)}\alpha(t)) |\partial_x \xi^{(j)}(x,t)|\, dx \\ &= \sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} \partial_\alpha \sigma(\Delta^{(j)}\alpha(t)) \partial_t (\Delta^{(j)}\alpha(t)) - (V_t^{(j)})^2 + \sigma(\Delta^{(j)}\alpha(t)) \partial_s \lambda_t^{(j)}\, ds. \end{aligned} \tag{4.2}$$

It follows from the definition of $\Delta^{(j)}\alpha = \alpha^{(j-1)} - \alpha^{(j)}$ and (1.4) that

$$\begin{aligned} &\sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} \partial_\alpha \sigma(\Delta^{(j)}\alpha(t)) \partial_t (\Delta^{(j)}\alpha(t))\, ds \\ &= \sum_{j=1}^{3} L^{(j)}(t) \partial_\alpha \sigma(\Delta^{(j)}\alpha(t)) (\partial_t \alpha^{(j-1)}(t) - \partial_t \alpha^{(j)}(t)) \\ &= \sum_{j=1}^{3} \left(\partial_\alpha \sigma(\Delta^{(j+1)}\alpha(t)) L^{(j+1)}(t) - \partial_\alpha \sigma(\Delta^{(j)}\alpha(t)) L^{(j)}(t) \right) \partial_t \alpha^{(j)}(t) \\ &= -\sum_{j=1}^{3} \frac{(\partial_t \alpha^{(j)}(t))^2}{\gamma}. \end{aligned} \tag{4.3}$$

We also have by (1.6), (2.5) and $\lambda_t^{(j)} = \langle \partial_t \vec{a}, \tau_t^{(j)} \rangle$ at the junction point $\vec{a}$

$$\sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} \sigma(\Delta^{(j)}\alpha(t)) \partial_s \lambda_t^{(j)}\, ds = \sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha(t)) \langle \partial_t \vec{a}, \tau_t^{(j)} \rangle \lfloor_{\text{at } \vec{a}} = 0. \tag{4.4}$$

Combining (4.2)–(4.4), we obtain (4.1). ◻

We next give a sufficient condition to obtain the boundedness of $L^{(j)}$ from below.

Lemma 4.2. *Assume (A1)–(A3). Then, there exist constants* $L_{\min} > 0$ *and* $m_3 > 0$ *such that*

$$L^{(j)}(t) \geq L_{\min} \quad \text{for} \quad j \in \{1,2,3\}, \ \ t \in [0,T)$$

if $E(0) \leq \sigma(0) m_3$.

Proof. First, we introduce a functional $\tilde{E}(\vec{a})$ for $\vec{a} \in \mathbb{R}^2$ as

$$\tilde{E}(\vec{a}) := |\vec{a} - P^{(1)}| + |\vec{a} - P^{(2)}| + |\vec{a} - P^{(3)}|.$$

Let $\vec{a}_0$ be the Fermat point of the triangle generated by the boundary points $\{P^{(j)}\}_{j\in\{1,2,3\}}$, then $\vec{a}_0$ is the unique minimizer of $\tilde{E}$. Then, we can see that the following three properties hold; $\tilde{E}(\vec{a}) \to \infty$ as $|\vec{a}| \to \infty$; $\tilde{E}$ is continuous; $\vec{a}_0$ is an interior point in the triangle with vertexes $P^{(1)}, P^{(2)}, P^{(3)}$, in particular, $\vec{a}_0$ is away from $P^{(j)}$ for $j = 1,2,3$. Therefore, we can choose a constant $m_3 > \tilde{E}(\vec{a}_0)$ sufficiently close to $\tilde{E}(\vec{a}_0)$ so that

$$S_{m_3} := \{\vec{a} \in \mathbb{R}^2 : \tilde{E}(\vec{a}) \leq m_3\}$$

is contained in the triangle generated by $\{P^{(j)}\}_{j\in\{1,2,3\}}$. Let $L_{\min} > 0$ be the shortest of the three distances between S_{m_3} and $P^{(j)}$ for $j = 1,2,3$.

Since $\sigma(\alpha) \geq \sigma(0)$ for any $\alpha \in \mathbb{R}$ and the distance $|\vec{a}(t) - P^{(j)}|$ is not larger than the length $L^{(j)}(t)$, we can see

$$E(t) \geq \sigma(0)\tilde{E}(\vec{a}(t)).$$

Thus, by the monotonicity of $E(t)$ as in (4.1), we have

$$\tilde{E}(\vec{a}(t)) \leq E(0)/\sigma(0) \leq m_3$$

if $E(0) \leq \sigma(0) m_1$. By the choice of m_1, we can see that $\vec{a}(t) \in S_{m_1}$ and

$$L^{(j)}(t) \geq |\vec{a}(t) - P^{(j)}| \geq \operatorname{dist}(S_{m_3}, \{P^{(j)}\}_{j\in\{1,2,3\}}) = L_{\min}$$

for any $t \in [0,T)$. ◻

In order to prove the global existence of the geometric flow later, we have to prove that the boundary conditions of the linearized system of the geometric flow as in Section 2.2 around an arbitrary positive time $t > 0$ satisfy the complementing condition. According to Lemma 2.7, the complementarity of the boundary conditions are ensured if $\Theta^{(j+1)} - \Theta^{(j)} \in (0,\pi)$ holds at the junction point $\vec{a}$ for any positive time, where $\Theta^{(j)}$ is the angle function defined as in (2.1). Therefore, we next study the relation between the angles $\Theta^{(j+1)} - \Theta^{(j)}$ at the junction point $\vec{a}$ and the surface tensions $\sigma(\Delta^{(j)}\alpha(t))$.

Lemma 4.3. *For any geometric flow governed by* (1.3)–(1.7)*, the equality*

$$\langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle = \frac{(\sigma(\Delta^{(j-1)}\alpha(t)))^2 - (\sigma(\Delta^{(j)}\alpha(t)))^2 - (\sigma(\Delta^{(j+1)}\alpha(t)))^2}{2\sigma(\Delta^{(j)}\alpha(t))\sigma(\Delta^{(j+1)}\alpha(t))} \tag{4.5}$$

holds at the junction point $\vec{a}(t)$ *for* $j \in \{1,2,3\}$ *and* $t > 0$.

Proof. For simplicity, we write $\sigma^{(j)}$ as $\sigma(\Delta^{(j)}\alpha(t))$ for $j \in \{1,2,3\}$. Taking the inner product of (1.6) and $\tau_t^{(i)}$ for $i = 1,2,3$, we have

$$\begin{pmatrix} \sigma^{(2)} & 0 & \sigma^{(3)} \\ \sigma^{(1)} & \sigma^{(3)} & 0 \\ 0 & \sigma^{(2)} & \sigma^{(1)} \end{pmatrix} \begin{pmatrix} \langle \tau_t^{(1)}, \tau_t^{(2)} \rangle \\ \langle \tau_t^{(2)}, \tau_t^{(3)} \rangle \\ \langle \tau_t^{(3)}, \tau_t^{(1)} \rangle \end{pmatrix} = - \begin{pmatrix} \sigma^{(1)} \\ \sigma^{(2)} \\ \sigma^{(3)} \end{pmatrix}$$

at the junction point $\vec{a}(t)$. Thus, (4.5) can be obtained by calculating the inverse of the matrix. □

Remark 4.4. It immediately follows from Lemma 4.3 that

$$|\cos(\Theta^{(j+1)} - \Theta^{(j)})| = |\langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle| < 1$$

holds if and only if

$$(\sigma(\Delta^{(j)}\alpha(t)) - \sigma(\Delta^{(j+1)}\alpha(t)))^2 < (\sigma(\Delta^{(j-1)}\alpha(t)))^2 < (\sigma(\Delta^{(j)}\alpha(t)) + \sigma(\Delta^{(j+1)}\alpha(t)))^2 \tag{4.6}$$

holds for any $j \in \{1,2,3\}$ and $t \in [0,T]$. We thus see that a geometric flow governed by (1.3)–(1.7) and starting from $\{\Gamma_0^{(j)}\}_{j\in\{1,2,3\}}, \vec{\alpha}_0$ satisfying $\Theta_0^{(j+1)}(0) - \Theta_0^{(j)}(0) \in (0,\pi)$ preserves the property $\Theta^{(j+1)}(0,t) - \Theta^{(j)}(0,t) \in (0,\pi)$ for any $t \in [0,T]$ if and only if $\vec{\alpha}$ satisfies (4.6) for any $t \in [0,T]$.

We here also discuss the following formula of $\langle \tau_t^{(j)}, \nu_t^{(j+1)} \rangle$ to apply in the L^2 or higher order estimate of curvatures later.

Lemma 4.5. *For any geometric flow governed by* (1.3)–(1.7)*, if* $\Theta^{(j+1)} - \Theta^{(j)} \in (0,\pi)$ *at the junction point* $\vec{a}(t)$ *for* $j \in \{1,2,3\}$ *and* $t > 0$*, the equality*

$$\langle \tau_t^{(j)}, \nu_t^{(j+1)} \rangle = -\frac{\sqrt{\left\{\sum_{i<k} 2(\sigma(\Delta^{(i)}\alpha(t)))^2(\sigma(\Delta^{(k)}\alpha(t)))^2\right\} - \left\{\sum_{i=1}^3 (\sigma(\Delta^{(i)}\alpha(t)))^4\right\}}}{2\sigma(\Delta^{(j)}\alpha(t))\sigma(\Delta^{(j+1)}\alpha(t))} \tag{4.7}$$

holds at the junction point $\vec{a}(t)$ *for* $j \in \{1,2,3\}$ *and* $t > 0$*.*

Proof. Since the curves $\Gamma_t^{(j)}$ are numbered counter-clockwise around the junction point, we have by $\Theta^{(j+1)} - \Theta^{(j)} \in (0,\pi)$ at the junction point $\vec{a}(t)$

$$\begin{aligned}\langle \tau_t^{(j)}, \nu_t^{(j+1)} \rangle &= \cos(\Theta^{(j+1)} - \Theta^{(j)} + \pi/2) = -\sin(\Theta^{(j+1)} - \Theta^{(j)}) \\ &= -\sqrt{1 - \cos^2(\Theta^{(j+1)} - \Theta^{(j)})} = -\sqrt{1 - (\langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle)^2}\end{aligned}$$

due to the choice of the direction of $\tau_t^{(j)}$ and $\nu_t^{(j)}$. Substituting (4.5) into the above equality, we obtain

$$\begin{aligned}\langle \tau_t^{(j)}, \nu_t^{(j+1)} \rangle &= -\frac{\sqrt{4(\sigma^{(j)})^2(\sigma^{(j+1)})^2 - \{(\sigma^{(j-1)})^2 - (\sigma^{(j)})^2 - (\sigma^{(j+1)})^2\}^2}}{2\sigma^{(j)}\sigma^{(j+1)}} \\ &= -\frac{\sqrt{2\left\{(\sigma^{(j)})^2(\sigma^{(j+1)})^2 + (\sigma^{(j-1)})^2(\sigma^{(j)})^2 + (\sigma^{(j-1)})^2(\sigma^{(j+1)})^2\right\} - \left\{\sum_{i=1}^3 (\sigma^{(i)})^4\right\}}}{2\sigma^{(j)}\sigma^{(j+1)}},\end{aligned}$$

where $\sigma^{(i)} = \sigma(\Delta\alpha^{(i)}(t))$ for $i = 1,2,3$. This equality is equivalent to (4.7). □

Since the right hand side of (4.5) consists of only $\sigma(\Delta^{(j)}\alpha(t))$, we can estimate the angle $\Theta^{(j+1)}(0,t) - \Theta^{(j)}(0,t)$ by $\sigma(\Delta^{(j)}\alpha(t))$. We thus continue to estimate the orientation parameters $\alpha^{(j)}$. We note that the estimates of $\alpha^{(j)}$ will be also used to obtain L^2 or higher order estimate of the curvatures. We first prove the dissipation of the orientations and misorientations.

Lemma 4.6. *Assume (A1) and (A3). Then,*

$$\frac{d}{dt}\sum_{j=1}^3 (\alpha^{(j)}(t))^2 \leq 0, \tag{4.8}$$

$$\frac{d}{dt}\sum_{j=1}^{3}(\Delta^{(j)}\alpha(t))^2 \leq 0 \tag{4.9}$$

for any $t \in [0,T)$.

Proof. Multiplying (1.4) by $\alpha^{(j)}$ and taking the sum for $j = 1, 2, 3$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{3}(\alpha^{(j)}(t))^2 &= -\gamma\sum_{j=1}^{3}\left(\partial_\alpha\sigma(\Delta^{(j+1)}\alpha(t))L^{(j+1)}(t) - \partial_\alpha\sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t)\right)\alpha^{(j)}(t)\\ &= -\gamma\sum_{j=1}^{3}\left(\partial_\alpha\sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t)\right)(\alpha^{(j-1)}(t) - \alpha^{(j)}(t))\\ &= -\gamma\sum_{j=1}^{3}\Delta^{(j)}\alpha(t)\partial_\alpha\sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t) \leq 0.\end{aligned}$$

Here, the inequality $\sigma_\alpha(\alpha)\alpha \geq 0$ for $\alpha \in \mathbb{R}$ have been used, which follows from the assumption (A3). Thus, (4.8) holds.

We have by a simple calculation and (1.4)

$$\frac{d}{dt}\Delta^{(j)}\alpha(t) = \gamma\left\{-2\partial_\alpha\sigma(\Delta^{(j)}\alpha(t))L^{(j)}(t) + \partial_\alpha\sigma(\Delta^{(j-1)}\alpha(t))L^{(j-1)}(t) + \partial_\alpha\sigma(\Delta^{(j+1)}\alpha(t))L^{(j+1)}(t)\right\}$$

Multiplying the equality by $\Delta^{(j)}\alpha$ and taking the sum for $j = 1, 2, 3$ we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{3}(\Delta^{(j)}\alpha(t))^2 = -3\gamma\sum_{j=1}^{3}\Delta^{(j)}\alpha\partial_\alpha(\Delta^{(j)}\alpha(t))L^{(j)}(t) \leq 0. \tag{4.10}$$

Thus, (4.9) holds. □

The exponential decay of the misorientations and derivatives of the orientations or surface tensions can be obtained assuming additional condition (A2) and the smallness of the energy as follows.

Corollary 4.7. *Assume (A1)–(A3). Let* m_3 *be the constant defined in Lemma 4.2 and* m_4 *be an arbitrary positive constant. Then, there exist constants* $c_4, c_5, c_6 > 0$ *such that if* $E(0) \leq \sigma(0)m_3$ *and*

$$\sum_{j=1}^{3}(\Delta^{(j)}\alpha(0))^2 \leq m_4 \tag{4.11}$$

then,

$$\sum_{j=1}^{3}(\Delta^{(j)}\alpha(t))^2 \leq e^{-c_4 t}\sum_{j=1}^{3}(\Delta^{(j)}\alpha(0))^2 \quad \textit{for} \quad t \in [0,T), \tag{4.12}$$

$$|\partial_\alpha\sigma(\Delta^{(j)}\alpha(t))| \leq c_5 e^{-\frac{c_4 t}{2}}\sqrt{\sum_{j=1}^{3}(\Delta^{(j)}\alpha(0))^2} \quad \textit{for} \quad t \in [0,T), \quad j \in \{1,2,3\}, \tag{4.13}$$

$$|\partial_t\alpha^{(j)}(t)| \leq c_6 e^{-\frac{c_4 t}{2}}\sqrt{\sum_{j=1}^{3}(\Delta^{(j)}\alpha(0))^2} \quad \textit{for} \quad t \in [0,T), \quad j \in \{1,2,3\}. \tag{4.14}$$

Proof. First, we prove (4.12). From (4.9) and (4.11), we have $|\Delta^{(i)}\alpha(t)| \le m_4^{1/2}$ for $t > 0$ and $i \in \{1,2,3\}$. Due to the assumption (A3), we can see by the boundedness of $|\Delta^{(i)}\alpha(t)|$ and the Taylor expansion

$$\partial_\alpha \sigma(\Delta^{(j)}\alpha(t))\Delta^{(j)}\alpha(t) \ge \left(\min_{|\alpha|\le m_4^{1/2}} \partial_\alpha^2 \sigma(\alpha)\right)(\Delta^{(j)}\alpha(t))^2.$$

Note that $\min_{|\alpha|\le m_4^{1/2}} \partial_\alpha^2\sigma(\alpha)$ is positive due to the assumption (A3). Applying this inequality and Lemma 4.2 to (4.10), we obtain

$$\frac{d}{dt}\sum_{j=1}^3 (\Delta^{(j)}\alpha(t))^2 \le -c_4 \sum_{j=1}^3 (\Delta^{(j)}\alpha(t))^2$$

for some $c_4 > 0$ and hence $\sum_{j=1}^3 (\Delta^{(j)}\alpha(t))^2$ decreases exponentially.

Next, we prove (4.13) and (4.14). Due to (4.1) and $\sigma(\alpha) \ge \sigma(0)$, we have

$$E(0) \ge E(t) \ge \sigma(0)\sum_{j=1}^3 L^{(j)}(t). \tag{4.15}$$

It also follows from the boundedness of $|\Delta^{(j)}\alpha(t)|$, the Taylor expansion, the assumption (A3) and the inequality (4.12) that

$$|\partial_\alpha\sigma(\Delta^{(i)}\alpha(t))| \le \left(\max_{|\alpha|\le m_4^{1/2}} \partial_\alpha^2\sigma(\alpha)\right) e^{-\frac{c_4 t}{2}} \sqrt{\sum_{j=1}^3 (\Delta^{(j)}\alpha(0))^2} \tag{4.16}$$

for any $i \in \{1,2,3\}$. We thus obtain (4.13) by letting $c_5 := \max_{|\alpha|\le m_4^{1/2}} \partial_\alpha^2\sigma(\alpha)$. Applying (4.15) and (4.16) to (1.4), we obtain (4.14). □

Remark 4.8. The boundedness of $\alpha^{(j)}$ and $\Delta^{(j)}\alpha$ can be obtained assuming only (A1) and (A3) since the boundedness immediately follows from Lemma 4.6. Even if one wants to get only the boundedness of $\partial_\alpha\sigma(\Delta^{(j)}\alpha)$ and $\partial_t\alpha^{(j)}$, but not the exponential decay as in Corollary 4.7, the assumptions (A2) and $E(0) \le \sigma(0)m_4$ are not necessary.

Due to the exponential decay of the misorientations, as shown in the following lemma, we can see the exponential stability of the angle condition of the Steiner triod at the junction point. In other word, $\Theta^{(j+1)} - \Theta^{(j)}$ exponentially converges to $2\pi/3$ at the junction point if the initial misorientations are sufficiently small.

Lemma 4.9. *Assume (A1)–(A3). Let m_3 be the constant in Lemma 4.2. Then, there exist $\varepsilon_1, c_7 > 0$ such that if $E(0) \le \sigma(0)m_3$ and*

$$\sum_{j=1}^3 (\Delta^{(j)}\alpha(0))^2 \le \varepsilon_1,$$

then the unit tangent vectors $\tau_t^{(i)}$ and $\tau_t^{(j)}$ of the geometric flow satisfy

$$\left|\langle \tau_t^{(i)}, \tau_t^{(j)}\rangle + \frac{1}{2}\right| \le c_7 e^{-\frac{c_4}{2}t}\sqrt{\sum_{j=1}^3 (\Delta^{(j)}\alpha(0))^2} \quad \text{for vary } i,j \in \{1,2,3\},\ \ t \in [0,T) \tag{4.17}$$

at the junction point $\vec{a}$, where c_4 is the constant defined in Corollary 4.7 replaced m_4 by ε_1.

Proof. Let

$$\tilde{\varepsilon} := \sqrt{\sum_{j=1}^{3} (\Delta^{(j)}\alpha(0))^2}$$

in this proof for simplicity. By (4.11) and (4.12), we have

$$|\sigma(0) - \sigma(\Delta^{(j)}\alpha(t))| \le \max_{|\alpha| \le \varepsilon_1^{1/2}} |\partial_\alpha \sigma(\alpha)||\Delta^{(j)}\alpha(t)| \le Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon} \tag{4.18}$$

for any j and some $C > 0$. We thus obtain by (4.5), (4.18) and $\tilde{\varepsilon} \le \varepsilon_1^{1/2}$

$$\begin{aligned}
&- \langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle \\
&= \frac{(\sigma(\Delta^{(j)}\alpha(t)))^2 + (\sigma(\Delta^{(j+1)}\alpha(t)))^2 - (\sigma(\Delta^{(j-1)}\alpha(t)))^2}{2\sigma(\Delta^{(j)}\alpha(t))\sigma(\Delta^{(j+1)}\alpha(t))} \\
&\ge \frac{2(\sigma(0) - Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon})^2 - (\sigma(0) + Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon})^2}{2(\sigma(0) + Ce^{-\frac{c_4}{2}t}\varepsilon_1^{1/2})^2} \\
&= \frac{2\{(\sigma(0) + Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon}) - 2Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon}\}^2 - (\sigma(0) + Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon})^2}{2(\sigma(0) + Ce^{-\frac{c_4}{2}t}\varepsilon_1^{1/2})^2} \\
&= \frac{(\sigma(0) + Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon})^2 - 8(\sigma(0) + Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon})Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon} + 8(Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon})^2}{2(\sigma(0) + Ce^{-\frac{c_4}{2}t}\varepsilon_1^{1/2})^2} \\
&\ge \frac{1}{2} - \frac{4Ce^{-\frac{c_4}{2}t}\tilde{\varepsilon}}{\sigma(0) + Ce^{-\frac{c_4}{2}t}\varepsilon_1} + \frac{4(Ce^{-\frac{c_4}{2}t})^2\tilde{\varepsilon}}{(\sigma(0) + Ce^{-\frac{c_4}{2}t}\varepsilon_1^{1/2})^2} \\
&\ge \frac{1}{2} - C'e^{-\frac{c_4}{2}t}\tilde{\varepsilon},
\end{aligned}$$

where $C' > 0$ depends only on $\sigma(0)$ and C (we may choose ε_1 small so that $C' > 0$ if necessary). Therefore we have $\langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle \le -\frac{1}{2} + C'e^{-\frac{c_4}{2}t}\tilde{\varepsilon}$. Similarly we can obtain $\langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle \ge -\frac{1}{2} - C''e^{-\frac{c_4}{2}t}\tilde{\varepsilon}$ for some constant $C'' > 0$. □

Note that Lemma 4.9 also implies that $\Theta_0^{(j+1)} - \Theta_0^{(j)} \in (0, \pi)$ holds at the junction point $\vec{a}(0)$ if (1.6) at $t = 0$ are satisfied and the initial misorientations is sufficiently small. Although we already obtained a sufficient condition to obtain that $|\langle \tau_t^{(i)}, \tau_t^{(j)} \rangle|$ is uniformly less than 1 at the junction point as in Lemma 4.9, this kind of the boundedness will be applied also to the L^2 or higher order estimate of the curvatures that . We thus note it as the following corollary to make it easier to cite.

Corollary 4.10. *Assume (A1)–(A3). Let m_3 be the constant in Lemma 4.2 and $m_5 \in (1/2, 1)$ be arbitrary. Then, there exists ε_2 such that if $E(0) \le \sigma(0)m_3$ and*

$$\sum_{j=1}^{3} (\Delta^{(j)}\alpha(0))^2 \le \varepsilon_2, \tag{4.19}$$

then the unit tangent vectors $\tau_t^{(i)}$ and $\tau_t^{(j)}$ of the geometric flow satisfy

$$\left| \langle \tau_t^{(i)}, \tau_t^{(j)} \rangle \right| \le m_5 \quad \text{for vary } i, j, k \in \{1, 2, 3\}, \quad t \in [0, T) \tag{4.20}$$

at the junction point $\vec{a}$.

5. Exponential L^2-decay of the curvatures

In this section, we prove the exponential L^2-decay of the curvatures assuming the closeness of the family of the initial datum to the equilibrium, namely, assuming smallness of the misorientations and the L^2-norm of the curvatures. The exponential decay will be applied to obtain higher order estimates of the curvatures which correspond to a smoothing effect and it also ensures the local exponential stability of the equilibrium in C^∞-topology.

We assume that a smooth geometric flow governed by (1.3)–(1.7) exists until a time $T > 0$ as in Section 4. We note that the restriction (2.9) is not assumed also in this section. The basic idea for the L^2-estimate is based on the energy method as in [15]. In order to obtain energy type inequality, we use the geometric identities in Lemma 2.1 and Lemma 2.2, and also additional identities as follows.

Lemma 5.1. *Any smooth geometric flow governed by* (1.3)–(1.7) *fulfills the following identities.*

$$\partial_t \tau_t^{(j)} = (\sigma(\Delta^{(j)}\alpha(t))\partial_s \kappa_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)})\nu_t^{(j)}, \tag{5.1}$$

$$\partial_t \kappa_t^{(j)} = \sigma(\Delta^{(j)}\alpha(t))\partial_s^2 \kappa_t^{(j)} + \sigma(\Delta^{(j)}\alpha(t))(\kappa_t^{(j)})^3 + \lambda^{(j)}\partial_s \kappa_t^{(j)} \tag{5.2}$$

for any $(x,t) \in [0,1] \times [0,T)$ *and* $j \in \{1,2,3\}$.

The proof is standard and thus we refer to [8, Chapter 1] for the details of the proof. An additional boundary identity at the junction point will be needed to control the boundary terms in the energy type inequality.

Lemma 5.2. *For any smooth geometric flow governed by* (1.3)–(1.7)*, there exists a smooth function* $f^{(j)} : [0,T) \to \mathbb{R}$ *for* $j \in \{1,2,3\}$ *such that*

$$\begin{aligned} &(\sigma(\Delta^{(j)}\alpha(t))\partial_s \kappa_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)}) - (\sigma(\Delta^{(j+1)}\alpha(t))\partial_s \kappa_t^{(j+1)} + \kappa_t^{(j+1)}\lambda_t^{(j+1)}) \\ &= f^{(j)}(\sigma(\Delta^{(1)}\alpha(t)), \sigma(\Delta^{(2)}\alpha(t)), \sigma(\Delta^{(3)}\alpha(t))) \end{aligned} \tag{5.3}$$

at the junction point $\vec{a}(t)$*. Furthermore, under the assumptions (A1)–(A3), there exist* $\varepsilon_3, c_8 > 0$ *such that if* $E(0) \le \sigma(0)m_3$*, where* m_3 *is the constant in Lemma 4.2, and*

$$\sum_{j=1}^{3} \left(\Delta^{(j)}\alpha_0\right)^2 \le \varepsilon_3, \tag{5.4}$$

then

$$|f^{(j)}(\sigma(\Delta^{(1)}\alpha(t)), \sigma(\Delta^{(2)}\alpha(t)), \sigma(\Delta^{(3)}\alpha(t)))| \le c_8 e^{-\frac{c_4}{2}t} \sqrt{\sum_{j=1}^{3} \left(\Delta^{(j)}\alpha_0\right)^2} \tag{5.5}$$

for any $j \in \{1,2,3\}$ *and* $t \in [0,T)$*, where* c_4 *is the constant defined in Corollary 4.7 replaced* m_4 *by* ε_3.

Proof. Let $\sigma^{(j)} = \sigma(\Delta^{(j)}\alpha(t))$ for simplicity. Due to (5.1) at the junction point, we have for any $j \in \{1,2,3\}$

$$\begin{aligned} \partial_t \langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle &= (\sigma^{(j)}\partial_s \kappa_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)})\langle \nu_t^{(j)}, \tau_t^{(j+1)} \rangle + (\sigma^{(j+1)}\partial_s \kappa_t^{(j+1)} + \kappa_t^{(j+1)}\lambda_t^{(j+1)})\langle \tau_t^{(j)}, \nu_t^{(j+1)} \rangle \\ &= \left((\sigma^{(j)}\partial_s \kappa_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)}) - (\sigma^{(j+1)}\partial_s \kappa_t^{(j+1)} + \kappa_t^{(j+1)}\lambda_t^{(j+1)}) \right) \langle \nu_t^{(j)}, \tau_t^{(j+1)} \rangle. \end{aligned}$$

We can see that $\Theta^{(j+1)} - \Theta^{(j)} \in (0, \pi)$ at the function point $\vec{a}$ for any $t > 0$ from Lemma 4.9 and, additionally, (4.20) also implies

$$\langle \nu_t^{(j)}, \tau_t^{(j+1)} \rangle = \sqrt{1 - \langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle^2} \geq \sqrt{1 - m_5^2} \tag{5.6}$$

for a fixed constant $m_5 \in (1/2, 1)$ if ε_3 is sufficiently small. Letting

$$f^{(j)}(\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}) := \frac{1}{\sqrt{1 - \langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle^2}} \partial_t \left(\frac{(\sigma^{(j-1)})^2 - (\sigma^{(j)})^2 - (\sigma^{(j+1)})^2}{2\sigma^{(j)}\sigma^{(j+1)}} \right), \tag{5.7}$$

we have (5.3) due to (4.5). The time derivative term can be formulated as

$$\partial_t \left(\frac{(\sigma^{(j-1)})^2 - (\sigma^{(j)})^2 - (\sigma^{(j+1)})^2}{2\sigma^{(j)}\sigma^{(j+1)}} \right) = \sum_{k=1}^{3} \frac{\hat{P}_k^{(j)}(\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)})}{\left(2\sigma^{(j)}\sigma^{(j+1)}\right)^2} \partial_\alpha \sigma(\Delta^{(k)}\alpha(t)) \partial_t \Delta^{(k)}\alpha(t), \tag{5.8}$$

where $\hat{P}_k^{(j)}$ are polynomials in $\sigma^{(1)}, \sigma^{(2)}$ and $\sigma^{(3)}$ with finite degree. Since $\sigma^{(k)}$ and $\partial_\alpha \sigma(\Delta^{(k)}\alpha(t))$ are bounded and $\sigma^{(k)} \geq \sigma(0)$, we thus have the boundedness of the absolute value of the coefficient of $\partial_t \Delta^{(k)}\alpha(t)$ in (5.8), which yields (5.5) due to (4.14) and (5.6). □

Another key to control boundary terms in the energy type inequality is the following estimate of the tangent velocity by the curvatures at the junction point, which enable us to derive an energy type inequality consists of only geometric values independent of change of parametrizations. The estimate of the tangent velocity can be obtained from Lemma 2.2 and Corollary 4.10 immediately.

Lemma 5.3. *Assume (A1)–(A3). Let $m_5 \in (1/2, 1)$ be an arbitrary constant. Let m_3 be the constant in Lemma 4.2 and ε_2 be the constant in Corollary 4.10. If $E(0) \leq \sigma(0)m_3$ and (4.19) holds, then*

$$|\lambda_t^{(j)}| \leq \frac{3}{1 - m_5^3} \sum_{i=1}^{3} \sigma(\Delta^{(i)}\alpha(t))|\kappa_t^{(i)}| \tag{5.9}$$

at the junction point for any $j \in \{1, 2, 3\}$ and $t \in (0, T)$.

We now derive an energy type equality from the geometric identities obtained above. The formulation and the calculation will be long, and thus we simplify the notions $\sigma(\Delta^{(j)}\alpha(t))$ and $\partial_\alpha \sigma(\Delta^{(j)}\alpha(t))$ to $\sigma^{(j)}$ and $\partial_\alpha \sigma^{(j)}$, respectively, for any $j \in \{1, 2, 3\}$.

Lemma 5.4. *Any smooth geometric flow fulfills the following energy type identity:*

$$\begin{aligned}
&\frac{d}{dt} \sum_{j=1}^{3} \int_{\Gamma^{(j)}} (\sigma^{(j)})^2 (\kappa_t^{(j)})^2 \, ds \\
&= \sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} (\sigma^{(j)})^3 \{-2(\partial_s \kappa_t^{(j)})^2 + (\kappa_t^{(j)})^4\} + 2\sigma^{(j)}(\partial_\alpha \sigma^{(j)})(\partial_t \Delta^{(j)}\alpha)(\kappa_t^{(j)})^2 \, ds \\
&\quad + \sum_{j=1}^{3} \left\{ \frac{2}{3} \left((\sigma^{(j)})^2 \kappa_t^{(j)} \right) \Big\lfloor_{at\, \vec{a}} f^{(j-1)} - \frac{2}{3} \left((\sigma^{(j)})^2 \kappa_t^{(j)} \right) \Big\lfloor_{at\, \vec{a}} f^{(j)} + \left((\sigma^{(j)})^2 (\kappa_t^{(j)})^2 \lambda_t^{(j)} \right) \Big\lfloor_{at\, \vec{a}} \right\}.
\end{aligned} \tag{5.10}$$

Proof. We have by (2.3), (2.5), (5.2) and integration by parts

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Gamma_t^{(j)}} (\sigma(\Delta^{(j)}\alpha))^2(\kappa_t^{(j)})^2\,ds\\
&= \int_{\Gamma_t^{(j)}} 2(\sigma^{(j)})^2\kappa_t^{(j)}\partial_t\kappa_t^{(j)} + 2\sigma^{(j)}(\partial_\alpha\sigma^{(j)})(\partial_t\Delta^{(j)}\alpha)(\kappa_t^{(j)})^2\\
&\quad + (\sigma^{(j)})^2(\kappa_t^{(j)})^2(-\sigma^{(j)}(\kappa_t^{(j)})^2 + \partial_s\lambda_t^{(j)})\,ds\\
&= \int_{\Gamma^{(j)}} (\sigma^{(j)})^3\{2\kappa_t^{(j)}\partial_s^2\kappa_t^{(j)} + (\kappa_t^{(j)})^4\} + 2\sigma^{(j)}(\partial_\alpha\sigma^{(j)})(\partial_t\Delta^{(j)}\alpha)(\kappa_t^{(j)})^2\\
&\quad + (\sigma^{(j)})^2\partial_s((\kappa_t^{(j)})^2\lambda_t^{(j)})\,ds\\
&= \int_{\Gamma^{(j)}} (\sigma^{(j)})^3\{-2(\partial_s\kappa_t^{(j)})^2 + (\kappa_t^{(j)})^4\} + 2\sigma^{(j)}(\partial_\alpha\sigma^{(j)})(\partial_t\Delta^{(j)}\alpha)(\kappa_t^{(j)})^2\,ds\\
&\quad + (\sigma^{(j)})^2(2\kappa_t^{(j)}\partial_s V_t^{(j)} + (\kappa_t^{(j)})^2\lambda_t^{(j)})\lfloor_{\text{at }\vec{a}}\\
&= \int_{\Gamma^{(j)}} (\sigma^{(j)})^3\{-2(\partial_s\kappa_t^{(j)})^2 + (\kappa_t^{(j)})^4\} + 2\sigma^{(j)}(\partial_\alpha\sigma^{(j)})(\partial_t\Delta^{(j)}\alpha)(\kappa_t^{(j)})^2\,ds\\
&\quad + 2\left((\sigma^{(j)})^2\kappa_t^{(j)}(\partial_s V_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)})\right)\Big\lfloor_{\text{at }\vec{a}} - \left((\sigma^{(j)})^2(\kappa_t^{(j)})^2\lambda_t^{(j)}\right)\Big\lfloor_{\text{at }\vec{a}}.
\end{aligned}
\tag{5.11}
$$

The equalities (2.6) and (5.3) imply

$$
\begin{aligned}
&\sum_{j=1}^{3}\left((\sigma^{(j)})^2\kappa_t^{(j)}(\partial_s V_t^{(j)} + \kappa_t^{(j)}\lambda_t^{(j)})\right)\Big\lfloor_{\text{at }\vec{a}}\\
&= \frac{1}{3}\sum_{j=1}^{3}\left\{\left((\sigma^{(j)})^2\kappa_t^{(j)}\right)\Big\lfloor_{\text{at }\vec{a}} f^{(j)}(\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}) - \left((\sigma^{(j)})^2\kappa_t^{(j)}\right)\Big\lfloor_{\text{at }\vec{a}} f^{(j-1)}(\sigma^{(1)},\sigma^{(2)},\sigma^{(3)})\right\}.
\end{aligned}
\tag{5.12}
$$

We here note that, for $A^{(j)}, B^{(j)}$ and $F^{(j)}$ with $\sum_{j=1}^{3} A_j = 0$ and $B^{(j)} = B^{(j+1)} + F^{(j)}$ for $j \in \{1,2,3\}$, an identity

$$
\begin{aligned}
\sum_{j=1}^{3} A^{(j)}B^{(j)} &= \frac{1}{3}(A^{(1)}B^{(1)} + A^{(2)}(B^{(1)} - F^{(1)}) + A^{(3)}(B^{(1)} + F^{(3)}))\\
&\quad + \frac{1}{3}(A^{(1)}(B^{(2)} + F^{(1)}) + A^{(2)}B^{(2)} + A^{(3)}(B^{(2)} - F^{(2)}))\\
&\quad + \frac{1}{3}(A^{(1)}(B^{(3)} - F^{(3)}) + A^{(2)}(B^{(3)} + F^{(2)}) + A^{(3)}B^{(3)}))\\
&= \frac{1}{3}\sum_{j=1}^{3} A^{(j)}(F^{(j)} - F^{(j-1)})
\end{aligned}
\tag{5.13}
$$

has been used. We thus have (5.10) by taking the summation of (5.11) with respect to $j \in \{1,2,3\}$. □

In order to obtain the exponential L^2-decay of the curvatures, our aim is to prove that the right hand side of (5.10) is bounded by sum of exponential decay term and the weighted L^2-norm of the curvature with negative coefficient. Therefore, the important part in (5.10) is the sum of the integration of $-2(\sigma^{(j)})^3(\partial_x\kappa_t^{(j)})^2$ appeared since a Poincaré type inequality can be applied to the term. Hereafter, we will introduce the Poincaré type inequality under the boundary conditions (2.5) and (2.6) by applying the Rayleigh quotient.

5.1. **Rayleigh quotient.** We consider the Rayleigh quotient in the following settings.

Definition 5.5. Let $\tilde{L}^{(j)}, \tilde{\sigma}^{(j)} > 0$ be fixed constants for $j \in \{1,2,3\}$ and define the Hilbert spaces H, V with inner products $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_V$ as

$$\begin{aligned} H &:= L^2(0, \tilde{L}^{(1)}) \times L^2(0, \tilde{L}^{(2)}) \times L^2(0, \tilde{L}^{(3)}), \\ V &:= \{\phi := (\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) \in H^1(0, \tilde{L}^{(1)}) \times H^1(0, \tilde{L}^{(2)}) \times H^1(0, \tilde{L}^{(3)}) : \\ &\qquad \phi^{(j)}(0) = 0, \quad (\tilde{\sigma}^{(1)})^2 \phi^{(1)}(\tilde{L}^{(1)}) + (\tilde{\sigma}^{(2)})^2 \phi^{(2)}(\tilde{L}^{(2)}) + (\tilde{\sigma}^{(3)})^2 \phi^{(3)}(\tilde{L}^{(3)}) = 0\}, \\ \langle \phi, \psi \rangle_H &:= \sum_{j=1}^{3} (\tilde{\sigma}^{(j)})^2 \int_0^{\tilde{L}^{(j)}} \phi^{(j)}(s) \psi^{(j)}(s)\, ds, \\ \langle \phi, \psi \rangle_V &:= \sum_{j=1}^{3} (\tilde{\sigma}^{(j)})^2 \left(\int_0^{\tilde{L}^{(j)}} \phi^{(j)}(s) \psi^{(j)}(s)\, ds + \int_0^{\tilde{L}^{(j)}} \partial_s \phi^{(j)}(s) \partial_s \psi^{(j)}(s)\, ds \right). \end{aligned}$$

The norm of H and V are denoted by $\| \cdot \|_H$ and $\| \cdot \|_V$, respectively.

We note that the boundary conditions in the Hilbert space V are from the boundary conditions (2.5) and (2.6) since we will substitute $\sigma(\Delta^{(j)} \alpha(t))$ and $\kappa_t^{(j)}$ for $\tilde{\sigma}^{(j)}$ and $\phi^{(j)}$, respectively. We next should introduce the following functionals to consider the Rayleigh quotient related to the negative term in (5.10).

Definition 5.6. We define a bilinear form $J : V \times V \to \mathbb{R}$ and a functional $I : V \setminus \{0\} \to \mathbb{R}$ by

$$\begin{aligned} J(\phi, \psi) &:= \sum_{j=1}^{3} (\tilde{\sigma}^{(j)})^3 \int_0^{\tilde{L}^{(j)}} \partial_s \phi^{(j)} \partial_s \psi^{(j)}\, ds, \\ I(\phi) &:= \frac{J(\phi, \phi)}{\langle \phi, \phi \rangle_H}. \end{aligned}$$

The following linear operator $\mathcal{L}$ is naturally leaded from the Euler-Lagrange equation of the Rayleigh quotient and we consider the weak form of $\mathcal{L}$.

Definition 5.7. For boundary conditions

$$\phi^{(j)}(0) = 0 \quad \text{for } j \in \{1,2,3\}, \tag{5.14}$$

$$\tilde{\sigma}^{(1)} \partial_s \phi^{(1)}(\tilde{L}^{(1)}) = \tilde{\sigma}^{(2)} \partial_s \phi^{(2)}(\tilde{L}^{(2)}) = \tilde{\sigma}^{(3)} \partial_s \phi^{(3)}(\tilde{L}^{(3)}), \tag{5.15}$$

$$(\tilde{\sigma}^{(1)})^2 \phi^{(1)}(\tilde{L}^{(1)}) + (\tilde{\sigma}^{(2)})^2 \phi^{(2)}(\tilde{L}^{(2)}) + (\tilde{\sigma}^{(3)})^2 \phi^{(3)}(\tilde{L}^{(3)}) = 0, \tag{5.16}$$

we define a linear operator $\mathcal{L}$ in H and its domain $\mathcal{D}(\mathcal{L})$ by

$$\begin{aligned} \mathcal{L}\phi &:= (\tilde{\sigma}^{(1)} \partial_s^2 \phi^{(1)}, \tilde{\sigma}^{(2)} \partial_s^2 \phi^{(2)}, \tilde{\sigma}^{(3)} \partial_s^2 \phi^{(3)}), \\ \mathcal{D}(\mathcal{L}) &:= \{\phi = (\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) : \\ &\qquad \phi^{(j)} \in H^2(0, \tilde{L}^{(j)}) \ \ (j \in \{1,2,3\}) \ \text{ and } \ \phi \ \text{ satisfy (5.14)–(5.16)}\}. \end{aligned}$$

Lemma 5.8. *All eigenvalues of $\mathcal{L}$ are nonzero real values.*

Proof. Since $\mathcal{L}$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_H$ and its weight is real, $\mathcal{L}$ has only real eigenvalues. We thus suppose by contradiction that $\mathcal{L}$ has a zero eigenvalue. Then, its eigenfunction $\phi_0 \in \mathcal{D}(\mathcal{L}) \setminus \{0\}$ satisfies $\phi_0^{(j)} \in C^\infty([0,\tilde{L}^{(j)}])$ and is of the form

$$\phi_0^{(j)}(s) = a^{(j)}s + b^{(j)} \quad \text{for } \ s \in [0,\tilde{L}^{(j)}],$$

where $a^{(j)}, b^{(j)} \in \mathbb{R}$, since $\partial_s^2\phi_0^{(j)} = 0$ in $[0,\tilde{L}^{(j)}]$ for $j \in \{1,2,3\}$. Applying the boundary conditions (5.14)–(5.16), we have

$$b^{(j)} = 0 \ \text{ for } \ j \in \{1,2,3\}, \quad \tilde{\sigma}^{(1)}a^{(1)} = \tilde{\sigma}^{(2)}a^{(2)} = \tilde{\sigma}^{(3)}a^{(3)}, \tag{5.17}$$

$$\sum_{j=1}^{3}(\tilde{\sigma}^{(j)})^2(a^{(j)}\tilde{L}^{(j)} + b^{(j)}) = 0. \tag{5.18}$$

Since $\tilde{\sigma}^{(j)}$ is positive, the equalities in (5.17) yield

$$\operatorname{sgn} a^{(j)} = \operatorname{sgn} a^{(i)} \quad \text{for } \ i,j \in \{1,2,3\},$$

where sgn is the signum function. It however contradicts (5.18) since $\tilde{L}^{(j)}$ is positive. □

Lemma 5.9. *The functional $I(\phi)$ has a minimizer $\overline{\phi} = (\overline{\phi}^{(1)}, \overline{\phi}^{(2)}, \overline{\phi}^{(3)}) \in V \setminus \{0\}$. Moreover, $\overline{\phi} \in \mathcal{D}(\mathcal{L})$, $\mathcal{L}\overline{\phi} = \zeta\overline{\phi}$, $\overline{\phi}^{(j)} \in C^\infty([0,\tilde{L}^{(j)}])$ for $j \in \{1,2,3\}$ and $\zeta < 0$, where $\zeta = -I(\overline{\phi})$.*

Proof. It is obvious that $I(\phi) \geq 0$ for any $\phi \in V \setminus \{0\}$. Therefore, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset V \setminus \{0\}$ such that

$$\lim_{n\to\infty} I(\phi_n) = \inf_{\phi\in V\setminus\{0\}} I(\phi).$$

We now put

$$\zeta := -\inf_{\phi\in V\setminus\{0\}} I(\phi).$$

Without loss of generality we may assume that $\|\phi_n\|_H = 1$ for $n \in \mathbb{N}$. Then

$$\|\phi_n\|_V \leq \left(\min_{j\in\{1,2,3\}} \tilde{\sigma}^{(j)}\right)^{-1} I(\phi) + 1$$

and hence $\{\phi_n\}_{n\in\mathbb{N}}$ is a bounded sequence in V. By similar discussions to those in [16, Section 8.12], we can show the existence of $\overline{\phi} \in V \setminus \{0\}$ such that $\phi_n \to \overline{\phi}$ in V and $I(\overline{\phi}) = -\zeta$.

Since $\overline{\phi}$ is a minimizer of I, for any $\psi \in V$ it holds that

$$\frac{dI(\overline{\phi} + t\psi)}{dt}\Big|_{t=0} = 0.$$

Therefore, we obtain

$$J(\overline{\phi},\psi) + \zeta\langle\overline{\phi},\psi\rangle_H = 0 \quad \text{for } \ \psi \in V. \tag{5.19}$$

Since $\overline{\phi}^{(j)} \in C^\infty([0,\tilde{L}^{(j)}])$ by a standard argument, it remains only to prove $\overline{\phi} \in \mathcal{D}(\mathcal{L})$ and $\zeta < 0$. Due to $\overline{\phi} \in V$, the minimizer $\overline{\phi}$ satisfies the boundary conditions (5.14) and (5.16). By the integration by parts for (5.19), we see that

$$\sum_{j=1}^{3}(\tilde{\sigma}^{(j)})^3\partial_s\overline{\phi}^{(j)}(\tilde{L}^{(j)})\psi^{(j)}(\tilde{L}^{(j)}) = 0 \quad \text{for } \ \psi \in V,$$

which yields (5.15) for $\overline{\phi}$ since ψ satisfies the boundary condition (5.16). It thus follows from Lemma 5.8 that $\zeta \neq 0$. On the other hand, $\zeta \leq 0$ is obvious due to the definition of I. Therefore, we have $\zeta < 0$. ◻

Since $-\zeta$ depends on $\tilde{L}^{(j)}$ and $\tilde{\sigma}^{(j)}$, we next show the continuity of $-\zeta$ in terms of $\tilde{L}^{(j)}$ and $\tilde{\sigma}^{(j)}$.

Lemma 5.10. *The minimum $-\zeta$ of I in $V \setminus \{0\}$ is continuous with respect to $\tilde{L}^{(j)}, \tilde{\sigma}^{(j)} > 0$ for $j \in \{1,2,3\}$.*

Proof. Fix $\tilde{L}_0^{(j)}, \tilde{\sigma}_0^{(j)} > 0$ and let $\tilde{L}^{(j)}, \tilde{\sigma}^{(j)} > 0$ be arbitrary constants to take the limit $\tilde{L}^{(j)} \to \tilde{L}_0^{(j)}$ and $\tilde{\sigma}^{(j)} \to \tilde{\sigma}_0^{(j)}$ for $j \in \{1,2,3\}$. The functional I_0 is defined as I in Definition 5.6 for $\tilde{L}_0^{(j)}, \tilde{\sigma}_0^{(j)}$ and let $-\zeta_0$ be its minimum in V replaced $\tilde{L}^{(j)}, \tilde{\sigma}^{(j)}$ by $\tilde{L}_0^{(j)}, \tilde{\sigma}_0^{(j)}$. We will thus prove $\zeta \to \zeta_0$ as $\tilde{L}^{(j)} \to \tilde{L}_0^{(j)}$ and $\tilde{\sigma}^{(j)} \to \tilde{\sigma}_0^{(j)}$ for $j \in \{1,2,3\}$.

Let $\phi \in V \setminus \{0\}$ be the minimizer of I and thus $I(\phi) = -\zeta$. We define $\phi_0 = (\phi_0^{(1)}, \phi_0^{(2)}, \phi_0^{(3)})$ as

$$\phi_0^{(j)}(\tilde{s}) := \frac{\tilde{\sigma}^{(j)}}{\tilde{\sigma}_0^{(j)}} \phi \left(\frac{\tilde{L}^{(j)}}{\tilde{L}_0^{(j)}} \tilde{s} \right) \quad \text{for } \tilde{s} \in [0, \tilde{L}_0^{(j)}],$$

and then the boundary conditions (5.14) and (5.16) replaced $\tilde{L}^{(j)}, \tilde{\sigma}^{(j)}$ by $\tilde{L}_0^{(j)}, \tilde{\sigma}_0^{(j)}$ are satisfied, and hence ϕ_0 is in V replaced $\tilde{L}^{(j)}, \tilde{\sigma}^{(j)}$ by $\tilde{L}_0^{(j)}, \tilde{\sigma}_0^{(j)}$. We also have by a simple calculation

$$\begin{aligned}
(\tilde{\sigma}_0^{(j)})^3 \int_0^{\tilde{L}_0^{(j)}} (\partial_{\tilde{s}} \phi_0^{(j)})^2 \, d\tilde{s} &= \frac{\tilde{\sigma}_0^{(j)} \tilde{L}^{(j)}}{\tilde{\sigma}^{(j)} \tilde{L}_0^{(j)}} \cdot (\tilde{\sigma}^{(j)})^3 \int_0^{\tilde{L}^{(j)}} (\partial_s \phi^{(j)})^2 \, ds, \\
(\tilde{\sigma}_0^{(j)})^2 \int_0^{\tilde{L}_0^{(j)}} (\phi_0^{(j)})^2 \, d\tilde{s} &= \frac{\tilde{L}_0^{(j)}}{\tilde{L}^{(j)}} \cdot (\tilde{\sigma}^{(j)})^2 \int_0^{\tilde{L}^{(j)}} (\phi^{(j)})^2 \, ds,
\end{aligned}$$

which implies by the definition of ζ_0

$$-\zeta_0 \leq I_0(\phi_0) \leq \left(\max_{j \in \{1,2,3\}} \frac{\tilde{\sigma}_0^{(j)} \tilde{L}^{(j)}}{\tilde{\sigma}^{(j)} \tilde{L}_0^{(j)}} \right) \left(\max_{j \in \{1,2,3\}} \frac{\tilde{L}^{(j)}}{\tilde{L}_0^{(j)}} \right) I(\phi) = - \left(\max_{j \in \{1,2,3\}} \frac{\tilde{\sigma}_0^{(j)} \tilde{L}^{(j)}}{\tilde{\sigma}^{(j)} \tilde{L}_0^{(j)}} \right) \left(\max_{j \in \{1,2,3\}} \frac{\tilde{L}^{(j)}}{\tilde{L}_0^{(j)}} \right) \zeta.$$

We can similarly obtain

$$-\zeta \leq - \left(\max_{j \in \{1,2,3\}} \frac{\tilde{\sigma}^{(j)} \tilde{L}_0^{(j)}}{\tilde{\sigma}_0^{(j)} \tilde{L}^{(j)}} \right) \left(\max_{j \in \{1,2,3\}} \frac{\tilde{L}_0^{(j)}}{\tilde{L}^{(j)}} \right) \zeta_0.$$

We thus see $\zeta \to \zeta_0$ as $\tilde{L}^{(j)} \to \tilde{L}_0^{(j)}$ and $\tilde{\sigma}^{(j)} \to \tilde{\sigma}_0^{(j)}$ for $j \in \{1,2,3\}$. ◻

We are now in a position to show the following Poincaré type inequality applying the Rayleigh quotient to the geometric flow.

Proposition 5.11. *Assume (A1)–(A3). Let a pair of $\{\Gamma_t^{(j)}\}_{j \in \{1,2,3\}}$ and $\vec{\alpha}$ be a smooth geometric flow governed by* (1.3)–(1.7) *in the time interval* $[0,T)$. *Let also $m_3 > 0$ be the constant in Lemma 4.2. Assume $E(0) \leq \sigma(0) m_3$. Then, there exists a constant $c_9 > 0$ depending only on m_3 and $\sum_{j=1}^3 (\sigma(\Delta^{(j)} \alpha(0)))^2$ such that*

$$c_9 \sum_{j=1}^3 \int_{\Gamma^{(j)}} (\sigma(\Delta^{(j)} \alpha(t)))^2 (\kappa_t^{(j)})^2 \, ds \leq \sum_{j=1}^3 \int_{\Gamma^{(j)}} (\sigma(\Delta^{(j)} \alpha(t)))^3 (\partial_s \kappa_t^{(j)})^2 \, ds \quad \textit{for } t \in [0,T). \tag{5.20}$$

Proof. Due to (4.1) and (4.9), we have by the assumption

$$L_{\min} \le L^{(j)}(t) \le m_3, \quad \sigma(0) \le \sigma(\Delta^{(j)}\alpha(t)) \le \sqrt{\sum_{j=1}^{3}(\sigma(\Delta^{(j)}\alpha(0)))^2}$$

for any $j \in \{1,2,3\}$ and $t \in [0,T)$, where $L_{\min}$ is the constant in Lemma 4.2. We thus have (5.20) by letting

$$c_9 := \min\left\{-\zeta = \inf_{\phi\in V\setminus\{0\}} I(\phi) : L_{\min} \le \tilde{L}^{(j)} \le m_3, \ \ \sigma(0) \le \tilde{\sigma}^{(j)} \le \sqrt{\sum_{j=1}^{3}(\sigma(\Delta^{(j)}\alpha(0)))^2}\right\} > 0$$

since $-\zeta$ is continuous. □

5.2. **Proof of the exponential L^2-decay of the curvatures.** In this section, we prove the exponential L^2-decay of the curvature for the geometric flow using (5.10). The Poincaré type inequality stated in Proposition 5.11 will be applied to the negative term $\sum -2(\sigma^{(j)})^3(\partial_x\kappa_t^{(j)})^2$ in the right hand side of (5.10). In order to control the remined terms in the right hand side of (5.10), we recall the following inequalities in [15, Lemma 6.2]. We refer to [15] for the proof.

Lemma 5.12 ([15, Lemma 6.2])**.** *For a C^3-curve $\Gamma_t^{(j)}$, there exist c_{10} and c_{11} depending only on $L^{(j)}(t)$ and $1/L^{(j)}(t)$, respectively, such that*

$$\int_{\Gamma_t^{(j)}} (\kappa_t^{(j)})^4\, ds \le 2\left(L^{(j)}(t)\|\partial_s\kappa_t^{(j)}\|_{L^2}^2 + \frac{1}{L^{(j)}(t)}\|\kappa_t^{(j)}\|_{L^2}^2\right)\|\kappa_t^{(j)}\|_{L^2}^2,$$

$$\left|\kappa_t^{(j)}\lfloor_{s=0}\right|^3 \le c_{10}\|\kappa_t^{(j)}\|_{L^2}\|\partial_s\kappa_t^{(j)}\|_{L^2}^2 + c_{11}\|\kappa_t^{(j)}\|_{L^2}^3.$$

We can prove the exponential L^2-decay of the curvature for the geometric flow assuming the smallness of $\Delta^{(j)}\alpha(0)$ and the weighted L^2-norm of $\kappa_0^{(j)}$ due to the discussion so far.

Proposition 5.13. *Assume (A1)–(A3). Let a pair of $\{\Gamma_t^{(j)}\}_{j\in\{1,2,3\}}$ and $\vec{\alpha}$ be a smooth geometric flow governed by (1.3)–(1.7) in the time interval $[0,T)$. Let also m_3 be the constant in Lemma 4.2. Then, there exist small $\varepsilon_4 > 0$ and $c_{12} > 0$ such that if*

$$E(0) \le \sigma(0)m_3, \quad \sum_{j=1}^{3}\left(\Delta^{(j)}\alpha_0\right)^2 \le \varepsilon_4, \quad \sum_{j=1}^{3}\int_{\Gamma_0^{(j)}}\left(\sigma(\Delta^{(j)}\alpha_0)\right)^2\left(\kappa_0^{(j)}\right)^2 ds \le \varepsilon_4, \tag{5.21}$$

then

$$\begin{aligned}&\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma(\Delta^{(j)}\alpha(t))\right)^2\left(\kappa_t^{(j)}\right)^2 ds\\ &\le e^{-c_{12}t}\left\{\sum_{j=1}^{3}\int_{\Gamma_0^{(j)}}\left(\sigma(\Delta^{(j)}\alpha_0)\right)^2\left(\kappa_0^{(j)}\right)^2 ds + \frac{8c_8^{\frac{3}{2}}}{c_4}\left(\sum_{j=1}^{3}\left(\Delta^{(j)}\alpha_0\right)^2\right)^{\frac{3}{4}}\right\}\end{aligned} \tag{5.22}$$

for $t \in [0,T)$, where c_4 and c_8 are the constants defined in Lemma 5.2.

Proof. We simplify the notation $\sigma(\Delta^{(j)}\alpha(t))$ to $\sigma^{(j)}$ for $j \in \{1,2,3\}$ in this proof. Note that we have by Lemma 4.1, Lemma 4.2, (4.9) and the definition of E

$$L_{\min} \le L^{(j)}(t) \le m_3, \quad \sigma(0) \le \sigma^{(j)} \le \sigma\left(\varepsilon_4^{1/2}\right) \quad \text{for } t \in [0,T), \ \ j \in \{1,2,3\} \tag{5.23}$$

due to the first assumption in (5.21). We will choose ε_4 in (5.21) small so that Corollary 4.7, Lemma 5.2 and Lemma 5.3 with a fixed constant $m_5 \in (1/2, 1)$ can be applied. It then follows from Young's inequality, the estimate of $\sigma(\Delta^{(j)}\alpha(t))$ in (5.23) and (5.9) that

$$\begin{aligned}\sum_{j=1}^{3}(\sigma^{(j)})^2(\kappa_t^{(j)})^2|\lambda_t^{(j)}| &\le \frac{3}{1-m_5^3}\left(\sum_{j=1}^{3}(\sigma^{(j)})^2(\kappa_t^{(j)})^2\right)\left(\sum_{i=1}^{3}\sigma^{(i)}|\kappa_t^{(i)}|\right) \\ &\le M_1\sum_{j=1}^{3}|\kappa_t^{(j)}|^3\end{aligned} \tag{5.24}$$

at the junction point for some constant $M_1 > 0$. We note also that there exists a constant $M_2 > 0$ such that

$$\sum_{j=1}^{3}\left|\frac{2}{3}\left((\sigma^{(j)})^2\kappa_t^{(j)}\lfloor_{\mathrm{at}\,\vec{a}}\right)f^{(j-1)} - \frac{2}{3}\left((\sigma^{(j)})^2\kappa_t^{(j)}\lfloor_{\mathrm{at}\,\vec{a}}\right)f^{(j)}\right| \le M_2\sum_{j=1}^{3}|\kappa_t^{(j)}\lfloor_{\mathrm{at}\,\vec{a}}|^3 + \sum_{j=1}^{3}|f^{(j)}|^{3/2} \tag{5.25}$$

due to Young's inequality, where $f^{(j)}$ is the function appeared in Lemma 5.2.

We first prove that if ε_4 is small, then (5.22) holds in any time interval $[0,\tau]$ such that

$$\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma(\Delta^{(j)}\alpha(t))\right)^2\left(\kappa_t^{(j)}\right)^2 ds \le \varepsilon_4 + \frac{8c_8^{\frac{3}{2}}}{c_4}\varepsilon_4^{\frac{3}{4}} \quad \text{for} \quad t\in[0,\tau] \tag{5.26}$$

holds. Assume (5.26) holds for a positive constant ε_4 to be chosen later. From Corollary 4.7, (5.5), Lemma 5.4, Lemma 5.12, (5.23), (5.24), (5.25) and (5.26), we obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^2\left(\kappa_t^{(j)}\right)^2 ds \\ &\le (M(\varepsilon_4)-2)\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^3\left(\partial_s\kappa_t^{(j)}\right)^2 ds + M(\varepsilon_4)\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^2\left(\kappa_t^{(j)}\right)^2 ds \\ &\quad + 3c_8^{\frac{3}{2}}e^{-\frac{3c_4}{4}t}\left(\sum_{j=1}^{3}\left(\Delta^{(j)}\alpha_0\right)^2\right)^{\frac{3}{4}}\end{aligned} \tag{5.27}$$

in the time interval $[0,\tau]$, where $M\in C([0,\infty))$ is a positive function such that $M(\varepsilon_4)\to 0$ as $\varepsilon_4\to 0$. Note that we used $\inf_{\alpha\in\mathbb{R}}\sigma(\alpha) > 0$ in (5.27). Therefore, due to Proposition 5.11, we can choose ε_4 small to obtain

$$\begin{aligned}&(M(\varepsilon_4)-2\mu)\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^3\left(\partial_s\kappa_t^{(j)}\right)^2 ds + M(\varepsilon_4)\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^2\left(\kappa_t^{(j)}\right)^2 ds \\ &\le -c_9\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^2\left(\kappa_t^{(j)}\right)^2 ds.\end{aligned} \tag{5.28}$$

We fix such ε_4 and let

$$c_{12} := \min\left\{c_9, \frac{3c_4}{8}\right\}, \quad a := \frac{8c_8^{\frac{3}{2}}}{c_4}\left(\sum_{j=1}^{3}\left(\Delta^{(j)}\alpha_0\right)^2\right)^{\frac{3}{4}}.$$

We then obtain by (5.27) and (5.28)

$$\frac{d}{dt}\left(e^{c_{12}t}\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^2\left(\kappa_t^{(j)}\right)^2 ds + ae^{-\frac{3c_4}{8}t}\right)$$

$$\leq (c_{12}-c_9)e^{c_{12}t}\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\left(\sigma^{(j)}\right)^2\left(\kappa_t^{(j)}\right)^2 ds + \left\{3c_8^{\frac{3}{2}}\left(\sum_{j=1}^{3}\left(\Delta^{(j)}\alpha_0\right)^2\right)^{\frac{3}{4}} - \frac{3c_4}{8}a\right\}e^{-\frac{3c_4}{8}t} \leq 0,$$

which yields (5.22) in the time interval $[0,\tau]$.

We finally prove that the inequality (5.26) holds whenever the flow exists under the assumption (5.21). Due to the assumption and the continuity of $\|\kappa^{(j)}\|_{L^2}$, the inequality (5.26) holds a short time interval $[0,\tau]$. We then have (5.22) in the time interval $[0,\tau]$, which yields (5.26) without equality at time $t=\tau$ by virtue of the assumption (5.21). Therefore, applying the continuity of $\|\kappa^{(j)}\|_{L^2}$ again, we see that the inequality (5.26) holds in a time interval $[0,\tau+\varepsilon]$ for some small $\varepsilon>0$. The argument can be applied whenever the flow exists, and thus the proof is completed. □

6. Higher order decay of the curvatures

In this section, we derive higher order decay of the curvatures and the idea is based on the higher order energy method as in [34]. Therefore, we will need higher order geometric identities as in Section 5 and let a smooth geometric flow governed by (1.3)–(1.7) exists until a time $T>0$ also in this section. Since statements in lemmas and calculations will be long, we simplify the notions $\sigma(\Delta^{(j)}\alpha(t))$ and $\partial_\alpha^k\sigma(\Delta^{(j)}\alpha(t))$ to $\sigma^{(j)}$ and $\partial_\alpha^k\sigma^{(j)}$, respectively, for any $j\in\{1,2,3\}$ and $k\in\mathbb{N}$. We further introduce some non standard notions for the computations in the sequel. The notions are extension of those in [34] to be able to apply the system including the orientation parameters $\vec{\alpha}$.

Definition 6.1. We denote by $P_h(\partial_s^i\kappa_t^{(j)})$ a polynomial in $\kappa_t^{(j)},\cdots,\partial_s^i\kappa_t^{(j)}$ such that every monomial it contains is of the form

$$C\prod_{l=0}^{i}(\partial_s^l\kappa_t^{(j)})^{A_l}\quad\text{with}\quad\sum_{l=0}^{i}(l+1)A_l=h,$$

where C is a constant coefficient. We will write $P'_h(\partial_s^i\kappa_t^{(j)})$ if every monomial further satisfies $\sum_{l=0}^{i}A_l\geq 2$. We also denote by $P_h(\partial_s^i\vec{\kappa}_t)$ a polynomial in $\kappa_t^{(j)},\cdots,\partial_s^i\kappa_t^{(j)}$ with any $j\in\{1,2,3\}$, such that every monomial it contains is of the form

$$C\prod_{j=1}^{3}\prod_{l=0}^{i}(\partial_s^l\kappa_t^{(j)})^{A_l^{(j)}}\quad\text{with}\quad\sum_{j=1}^{3}\sum_{l=0}^{i}(l+1)A_l^{(j)}=h.$$

We will call h the geometric order of P_h for both polynomials. When we write $P_h(|\partial_s^i\kappa_t^{(j)}|)$ (resp. $P_{\leq h}(|\partial_s^i\kappa_t^{(j)}|)$) we mean a finite sum of terms like

$$C\prod_{l=0}^{i}|\partial_s^l\kappa_t^{(j)}|^{A_l}\quad\text{with}\quad\sum_{l=0}^{i}(l+1)A_l=h\quad\left(\text{resp. }\sum_{l=0}^{i}(l+1)A_l\leq h,\ \sum_{l=0}^{i}A_l>0\right),\tag{6.1}$$

where C is a constant coefficient and the exponents A_l are non-negative real values. Similarly, let $P_h(\|\partial_s^i\kappa_t^{(j)}\|)$ and $P_{\leq h}(\|\partial_s^i\kappa_t^{(j)}\|)$ be a finite sum of terms like (6.1) replaced $|\partial_s^l\kappa_t^{(j)}|$ by $\|\partial_s^l\kappa_t^{(j)}\|_{L^\infty}$. These notions will be used for $P_h(\partial_s^i\vec{\kappa}_t)$ as $P_h(|\partial_s^i\vec{\kappa}_t|)$, $P_{\leq h}(|\partial_s^i\vec{\kappa}_t|)$, $P_h(\|\partial_s^i\vec{\kappa}_t\|)$ and $P_{\leq h}(\|\partial_s^i\vec{\kappa}_t\|)$.

We next denote by $Q(\vec{\sigma})$ a polynomial in $\partial_\alpha^n \sigma^{(j)}$ with $n \in \mathbb{N} \cup \{0\}$, $j \in \{1,2,3\}$, and by $Q(\vec{\sigma}, \partial_t \Delta\vec{\alpha})$ a polynomial in $\partial_\alpha^n \sigma^{(j)}$ with $n \in \mathbb{N} \cup \{0\}$, $j \in \{1,2,3\}$ and also in $\partial_t \Delta^{(j)}\alpha$ with $j \in \{1,2,3\}$. For $i \in \mathbb{N}$, let $\hat{Q}_h(\vec{\sigma}, \partial_t^i \Delta\vec{\alpha})$ denote a polynomial such that every monomial it contains is of the form

$$C(\vec{\sigma}) \prod_{j=1}^{3} \prod_{l=1}^{i} (\partial_t^l \Delta^{(j)}\alpha)^{A_l^{(j)}} \quad \text{with} \quad \sum_{j=1}^{3} \sum_{l=1}^{i} l A_l^{(j)} = h,$$

where $C(\vec{\sigma})$ is a monomial in $\partial_\alpha^n \sigma^{(j)}$ with $n \in \mathbb{N} \cup \{0\}$ and $j \in \{1,2,3\}$. Let also $R(\vec{\Theta})$ be a finite sum of fractions such that every fraction is of the form

$$\frac{\hat{P}(c^{(1)}, c^{(2)}, c^{(3)}, s^{(1)}, s^{(2)}, s^{(3)})}{(1 - c^{(1)}c^{(2)}c^{(3)})^n},$$

where n is an natural number and $\hat{P}$ is a polynomial in $c^{(j)} = \cos(\Theta^{(j+1)} - \Theta^{(j)})\lfloor_{\text{at } \vec{a}}$ and $s^{(j)} = \sin(\Theta^{(j+1)} - \Theta^{(j)})\lfloor_{\text{at } \vec{a}}$. When we write multiplications of above notions, we mean a finite sum of the multiplication (e.g.: One example of $Q(\vec{\sigma})P_2(\partial_s \vec{\kappa}_t)$ is $\sigma^{(1)}(\kappa_t^{(1)})^2 + \sigma^{(2)}\partial_s \kappa_t^{(2)}$).

Remark 6.2. By means of the definitions, we may easily see that $P_h(\partial_t^i \kappa_t^{(j)}) \le P_h(|\partial_t^i \kappa_t^{(j)}|) \le P_h(\|\partial_t^i \kappa_t^{(j)}\|)$. A similar inequality holds also for $P_h(\partial_t^i \vec{\kappa}_t)$. The calculus rules as

$$P_{h_1}(\partial_s^{i_1} \kappa_t^{(j)}) \cdot P_{h_2}(\partial_s^{i_2} \kappa_t^{(j)}) = P_{h_1+h_2}(\partial_x^{\max\{i_1,i_2\}} \kappa_t^{(j)}),$$
$$\hat{Q}_{h_1}(\vec{\sigma}, \partial_t^{i_1} \Delta\vec{\alpha}) \cdot \hat{Q}_{h_2}(\vec{\sigma}, \partial_t^{i_2} \Delta\vec{\alpha}) = \hat{Q}_{h_1+h_2}(\vec{\sigma}, \partial_t^{\max\{i_1,i_2\}} \vec{\alpha}),$$

and

$$\partial_t \hat{Q}_h(\vec{\sigma}, \partial_t^i \Delta\vec{\alpha}) = \hat{Q}_{h+1}(\vec{\sigma}, \partial_t^{i+1} \Delta\vec{\alpha}) \tag{6.2}$$

will be useful to calculate in the sequel. Note that the polynomials Q is bounded due to Lemma 4.6 and Corollary 4.7 (according to Remark 4.8, (A1) is a sufficient assumption to obtain the boundedness of Q), while the estimate of $\hat{Q}_h(\partial_t^i \Delta\vec{\alpha})$ is now not obvious for $i \ge 2$ and will be considered. The boundedness of R follows from (4.20). Notice also that, (2.7) can be re-writen as

$$\lambda_t^{(j)}\lfloor_{\text{at } \vec{a}} = Q(\vec{\sigma})R(\vec{\Theta})P_1(\vec{\kappa}_t)\lfloor_{\text{at } \vec{a}} \tag{6.3}$$

and the above identity, (2.4) and (5.2) imply

$$\partial_t R(\vec{\Theta}) = Q(\vec{\sigma})R(\vec{\Theta})P_2(\partial_s \vec{\kappa}_t)\lfloor_{\text{at } \vec{a}}, \quad \partial_t \lambda_t^{(j)}\lfloor_{\text{at } \vec{a}} = \{\hat{Q}_1(\vec{\sigma}, \partial_t \Delta\vec{\alpha})R(\vec{\Theta})P_1(\vec{\kappa}_t) + Q(\vec{\sigma})R(\vec{\Theta})P_3(\partial_s^2 \vec{\kappa}_t)\}\lfloor_{\text{at } \vec{a}}. \tag{6.4}$$

We first list higher order geometric identities as follows.

Lemma 6.3. *Any smooth geometric flow governed by* (1.3)–(1.7) *fulfills the following identities.*

$$\partial_t \partial_s^n \kappa_t^{(j)} = \sigma^{(j)} \left(\partial_s^{n+2} \kappa_t^{(j)} + P'_{n+3}(\partial_s^n \kappa_t^{(j)}) \right) + \lambda_t^{(j)} \partial_s^{n+1} \kappa_t^{(j)} \tag{6.5}$$

for any $(x,t) \in [0,1] \times (0,T)$, $n \in \mathbb{N} \cup \{0\}$ *and* $j \in \{1,2,3\}$. *Furthermore,*

$$\partial_s^{2n} \kappa_t^{(j)} \Big\lfloor_{at\ P^{(j)}} = 0 \tag{6.6}$$

for any $n \in \mathbb{N} \cup \{0\}$ *and* $j \in \{1,2,3\}$.

Proof. Note that we can obtain by (2.3) and $\partial_t \partial_x = \partial_x \partial_t$

$$\partial_t \partial_s = \partial_t \frac{\partial_x}{|\partial_x \xi^{(j)}|} = \partial_s \partial_t + (\sigma^{(j)}(\kappa_t^{(j)})^2 - \partial_s \lambda_t^{(j)})\partial_s \tag{6.7}$$

on each $\Gamma_t^{(j)}$. The identity (6.5) can be proved inductively. When $n=0$, the identity (6.5) coincides with (5.2). Assuming that (6.5) holds for some $n \in \mathbb{N}\cup\{0\}$, due to (6.7), we obtain

$$\begin{aligned}\partial_t\partial_s^{n+1}\kappa_t^{(j)} &= \partial_s\partial_t\partial_s^n\kappa_t^{(j)} + (\sigma^{(j)}(\kappa_t^{(j)})^2 - \partial_s\lambda_t^{(j)})\partial_s^{n+1}\kappa_t^{(j)}\\ &= \partial_s\left(\sigma^{(j)}(\partial_s^{n+2}\kappa_t^{(j)} + P'_{n+3}(\partial_s^n\kappa_t^{(j)})) + \lambda_t^{(j)}\partial_s^{n+1}\kappa_t^{(j)}\right) + (\sigma^{(j)}(\kappa_t^{(j)})^2 - \partial_s\lambda_t^{(j)})\partial_s^{n+1}\kappa_t^{(j)}\\ &= \sigma^{(j)}\left(\partial_s^{n+3}\kappa_t^{(j)} + P'_{n+4}(\partial_s^{n+1}\kappa_t^{(j)})\right) + \lambda_t^{(j)}\partial_s^{n+2}\kappa_t^{(j)}\end{aligned}$$

which is (6.5) replaced n by $n+1$.

The boundary condition (6.6) also can be proved inductively. When $n=0$, the condition (6.6) is proved by (2.5). Assume that (6.6) holds for any $0\le n\le m$. We then obtain applying (6.5)

$$\partial_s^{2(m+1)}\kappa_t^{(j)} = \frac{1}{\sigma^{(j)}}\left(\partial_t\partial_s^{2m}\kappa_t^{(j)} - \lambda_t^{(j)}\partial_s^{2m+1}\kappa_t^{(j)}\right) + P'_{2m+3}(\partial_s^{2m}\kappa_t^{(j)}) = P'_{2m+3}(\partial_s^{2m}\kappa_t^{(j)})$$

since $\partial_t\partial_s^{2m}\kappa_t^{(j)} = 0$ can be obtained by taking time derivative of (6.6) with $n=m$ and $\lambda_t^{(j)}=0$ is already proved in (2.5). We now note that, for each monomial $C\prod(\partial_s^l\kappa_t^{(j)})^{A_l}$ in P'_{2m+3}, the exponent A_l is nonzero at least for one even l since $\sum(l+1)A_l = 2m+3$ is odd, which implies $P'_{2m+3}(\partial_s^{2m}\kappa_t^{(j)}) = 0$ due to (6.6) with $0\le n\le m$. We thus obtain the conclusion. □

We next introduce some identities to extend (2.6) and (5.3) to higher order boundary conditions at the junction point. Since we will take higher order time derivatives of the original boundary conditions, we will need some formulas of time derivatives of $\sigma^{(j)}, f^{(j)}, L^{(j)}$ and so on.

We here note a formula to make later calculations simple as follows.

Lemma 6.4. *For any smooth function $f(t)$ and non-negative integers $A_0,\cdots,A_i \in \mathbb{N}\cup\{0\}$ with $\sum_{l=0}^i(l+1)A_l = h$, a smooth geometric flow governed by (1.3)–(1.7) fulfills*

$$\begin{aligned}&\partial_t\int_{\Gamma_t^{(j)}} f(t)\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l}\,ds\\ &= \int_{\Gamma_t^{(j)}}\partial_t f(t)\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l} + \sigma^{(j)}f(t)P_{h+2}(\partial_s^{i+2}\kappa_t^{(j)})\,ds + f(t)\lambda_t^{(j)}\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l}\Big\rfloor_{at\,\vec{a}}.\end{aligned}\tag{6.8}$$

Proof. By means of (2.3) and (6.5), we have by a simple calculation

$$\begin{aligned}&\partial_t\int_{\Gamma_t^{(j)}} f(t)\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l}\,ds\\ &= \int_{\Gamma_t^{(j)}}\partial_t f(t)\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l} + f(t)\sum_{n=0}^i\left(\prod_{j\neq n}(\partial_s^l\kappa_t^{(j)})^{A_l}\right)A_n(\partial_s^n\kappa_t^{(j)})^{A_n-1}\partial_t\partial_s^n\kappa_t^{(j)}\\ &\qquad + f(t)\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l}(-\sigma^{(j)}(\kappa_t^{(j)})^2 + \partial_s\lambda_t^{(j)})\,ds\\ &= \int_{\Gamma_t^{(j)}}\partial_t f(t)\prod_{l=0}^i(\partial_s^l\kappa_t^{(j)})^{A_l} + \sigma^{(j)}f(t)\sum_{n=0}^i P_{h-(n+1)}(\partial_s^i\kappa)(\partial_s^{n+2}\kappa_t^{(j)} + P_{n+3}(\partial_s^n\kappa_t^{(j)}))\\ &\qquad + f(t)\sum_{n=0}^i\left(\prod_{j\neq n}(\partial_s^l\kappa_t^{(j)})^{A_l}\right)A_n(\partial_s^n\kappa_t^{(j)})^{A_n-1}\lambda_t^{(j)}\partial_s^{n+1}\kappa_t^{(j)}\end{aligned}$$

$$+ f(t)\prod_{l=0}^{i}(\partial_s^l\kappa_t^{(j)})^{A_l}\partial_s\lambda_t^{(j)} + f(t)P_{l+2}(\partial_s^i\kappa_t^{(j)})\ ds$$

$$= \int_{\Gamma_t^{(j)}} \partial_t f(t)\prod_{l=0}^{i}(\partial_s^l\kappa_t^{(j)})^{A_l} + \sigma^{(j)} f(t)P_{l+2}(\partial_s^{i+2}\kappa) + f(t)\partial_s\left(\prod_{l=0}^{i}(\partial_s^l\kappa_t^{(j)})^{A_l}\lambda_t^{(j)}\right)\ ds.$$

The boundary condition $\lambda_t^{(j)}\lfloor_{\text{at } P^{(j)}} = 0$ as in (2.5) can be applied to the last term to obtain (6.8). □

We next introduce a formula of time derivatives of $L^{(j)}$.

Lemma 6.5. *For any smooth geometric flow governed by* (1.3)–(1.7)*, the derivative of the length* $\partial_t^n L^{(j)}$*, for any* $n \in \mathbb{N}$ *and* $j \in \{1,2,3\}$*, is a finite sum of*

$$Q(\vec{\sigma}, \partial_t\Delta\vec{\alpha})R(\vec{\Theta})\prod_{k=1}^{3}\prod_{l=1}^{n}\left(\int_{\Gamma_t^{(k)}} P_{2l}(\partial_s^{2l-2}\kappa_t^{(k)})\ ds\right)^{A_l^{(k)}}\cdot\prod_{l'=1}^{n}\left(P_{2l'-1}(\partial_s^{2l'-2}\vec{\kappa}_t)\lfloor_{at\ \vec{a}}\right)^{B_{l'}}\cdot\prod_{k=1}^{3}(L^{(k)})^{C^{(k)}}, \tag{6.9}$$

where $A_l^{(k)}, B_{l'}$ *and* $C^{(k)}$ *are non-negative integers satisfying*

$$\sum_{k=1}^{3}\sum_{l=1}^{n}A_l^{(k)} + \sum_{l'=1}^{n}B_{l'} \geq 1, \quad \sum_{k=1}^{3}\sum_{l=1}^{n}2lA_l^{(k)} + \sum_{l'=1}^{n}(2l'-1)B_{l'} \leq 2n.$$

Remark 6.6. The boundedness of $L^{(j)}$ can be obtained from Lemma 4.1. We also remind that the boundedness of $\sigma^{(j)}$ and $\partial_t\Delta^{(j)}\alpha$ can be obtained from Lemma 4.6 and Lemma 4.7. Therefore, applying these boundednesses to Lemma 6.5, we obtain

$$|\partial_t^n L^{(j)}| \leq P_{\leq 2n}(\|\partial_s^{2n-2}\vec{\kappa}_t\|) \tag{6.10}$$

for any $j \in \{1,2,3\}$, and thus $\sum 2lA_l^{(k)} + \sum(2l'-1)B_{l'}$ corresponds to the geometric order.

Proof. We prove inductively. We obtain by a simple calculation as in the proof of Lemma 4.1

$$\partial_t L^{(j)} = -\int_{\Gamma_t^{(j)}} \sigma^{(j)}(\kappa_t^{(j)})^2\ ds + \lambda_t^{(j)}\lfloor_{\text{at } \vec{a}} \tag{6.11}$$

and thus the claim holds for $n = 1$ due to (6.3). We now assume Lemma 6.5 holds for some $n \in \mathbb{N}$. We can see by (1.4), (2.4) and (6.3)

$$\partial_t\left(Q(\vec{\sigma}, \partial_t\Delta\vec{\alpha})R(\vec{\Theta})\right) = Q(\vec{\sigma}, \partial_t\Delta\vec{\alpha})R(\vec{\Theta})\left(1 + \sum_{k=1}^{3}(\partial_t^2\Delta^{(k)}\alpha + \partial_t\Theta^{(k)}\lfloor_{\text{at } \vec{a}})\right)$$

$$= Q(\vec{\sigma}, \partial_t\Delta\vec{\alpha})R(\vec{\Theta})\left(1 + \sum_{k=1}^{3}((L^{(k)})^2 + \partial_t L^{(k)} + P_2(\partial_s\vec{\kappa}_t)\lfloor_{\text{at } \vec{a}})\right)$$

$$= Q(\vec{\sigma}, \partial_t\Delta\vec{\alpha})R(\vec{\Theta})\left(1 + \sum_{k=1}^{3}\left((L^{(k)})^2 + \int_{\Gamma_t^{(k)}}(\kappa_t^{(k)})^2\ ds\right) + P_2(\partial_s\vec{\kappa}_t)\right),$$

which implies that, if we take time derivative of $Q(\vec{\sigma}, \partial_t\Delta\vec{\alpha})R(\vec{\Theta})$ in (6.9), the entire product is a finite sum of (6.9) replaced n by $n+1$. Notice also that, since $\sum 2lA_l^{(k)} + \sum(2l'-1)B_{l'}$ corresponds to the geometric order of the entire product (6.9), the above identity implies that $\sum 2lA_l^{(k)} + \sum(2l'-1)B_{l'}$

increase at most 2 if we take time derivative of $Q(\vec{\sigma}, \partial_t \Delta\vec{\alpha})R(\vec{\Theta})$ in (6.9). Similarly, we obtain by (6.8)

$$\partial_t \int_{\Gamma_t^{(k)}} P_{2l}(\partial_s^{2l-2}\kappa_t^{(k)})\, ds = \int_{\Gamma_t^{(k)}} \sigma^{(k)} P_{2l+2}(\partial_s^{2l}\kappa_t^{(k)})\, ds + Q(\vec{\sigma}, \partial_t \Delta\vec{\alpha})R(\vec{\Theta})P_{2l+1}(\partial_s^{2l-2}\vec{\kappa}_t)\lfloor_{\text{at } \vec{a}}$$

and by (6.5)

$$\partial_t P_{2l'-1}(\partial_s^{2l'-2}\vec{\kappa}_t)\lfloor_{\text{at } \vec{a}} = Q(\vec{\sigma}, \partial_t \Delta\vec{\alpha})R(\vec{\Theta})P_{2l'+1}(\partial_s^{2l'}\vec{\kappa}_t)\lfloor_{\text{at } \vec{a}}.$$

We thus, applying (6.11), Lemma 6.5 holds replaced n by $n+1$. We here note that the increasing rate of the geometric order of $P_{2l}, P_{2l'-1}$ and the highest differential order of $\partial_s^{2l-2}\kappa_t^{(k)}, \partial_s^{2l'-2}\vec{\kappa}_t$ can be seen from the last two identities. Notice also the increasing rate related to the geometric order of each part is at most 2 per time derivative, and thus we have $\sum_{l=1}^{n} 2lA_l^{(k)} + \sum_{l'=1}^{n}(2l'-1)B_{l'} \leq 2n$. □

We next consider the higher order derivatives of $\alpha^{(j)}$.

Lemma 6.7. *For any smooth geometric flow governed by* (1.3)–(1.7)*, the derivative of the misorientation* $\partial_t^n \Delta^{(j)}\alpha$*, for any* $n \in \mathbb{N}$ *and* $j \in \{1,2,3\}$*, is a finite sum of*

$$\prod_{j=1}^{3}\left(\prod_{l=0}^{n-1}(\partial_t^l L^{(j)})^{A_l^{(j)}}\right) Q(\vec{\sigma}) \quad \textit{with} \quad \sum_{j=1}^{3}\sum_{l=0}^{n-1}(l+1)A_l^{(j)} = n, \tag{6.12}$$

where $\hat{P}(\sigma^{(j)})$ *is a polynomial in* $\partial_\alpha^m \sigma^{(k)}$ *with* $m \in \mathbb{N}\cup\{0\}$ *and* $k \in \{1,2,3\}$ *such that every monomial it contains is non-constant.*

Proof. For $n = 1$, the claim holds by means of (1.4). The differential equation (1.4) implies also

$$\begin{aligned}
\partial_t \left(\prod_{l=0}^{n-1}(\partial_t^l L^{(j)})^{A_l^{(j)}}\right) Q(\vec{\sigma}) &= \sum_{m=0}^{n-1}\left(\prod_{l\neq m}(\partial_t^l L^{(j)})^{A_l^{(j)}}\right) \cdot A_m (\partial_t^m L^{(j)})^{A_m^{(j)}-1}\partial_t^{m+1}L^{(j)} \cdot Q(\vec{\sigma}) \\
&\quad + \left(\prod_{l=0}^{n-1}(\partial_t^l L^{(j)})^{A_l^{(j)}}\right) \cdot Q(\vec{\sigma}) \sum_{k=1}^{3} \partial_t \Delta^{(j)}\alpha \\
&= \sum_{m=0}^{n-1}\left(\prod_{l\neq m}(\partial_t^l L^{(j)})^{A_l^{(j)}}\right) \cdot A_m (\partial_t^m L^{(j)})^{A_m^{(j)}-1}\partial_t^{m+1}L^{(j)} \cdot Q(\vec{\sigma}) \\
&\quad + \left(\prod_{l=0}^{n-1}(\partial_t^l L^{(j)})^{A_l^{(j)}}\right) \cdot Q(\vec{\sigma}) \sum_{k=1}^{3} L^{(k)},
\end{aligned}$$

which shows that the form of each term is preserved in the sense of (6.12) and the increase rate of $\sum(l+1)A_l$ is just 1 per time derivative. We thus see that Lemma 6.7 replaced n by $n+1$ holds if it holds for some $n \in \mathbb{N}$. □

We now derive higher order boundary conditions from (2.6) and (5.3).

Lemma 6.8. *For any smooth geometric flow governed by* (1.3)–(1.7)*,* $n \in \mathbb{N}$ *and* $j \in \{1,2,3\}$*, there exist* $I_{2n}^{(j)}$ *and* $I_{2n+1}^{(j)}$ *such that*

$$\sum_{j=1}^{3}\left((\sigma^{(j)})^{n+2}\partial_s^{2n}\kappa_t^{(j)}\lfloor_{at\, \vec{a}} + I_{2n}^{(j)}\right) = 0, \tag{6.13}$$

$$\begin{aligned}(6.14)\qquad &(\sigma^{(j)})^{n+1}\partial_s^{2n+1}\kappa_t^{(j)}\lfloor_{at\,\vec{a}}+I_{2n+1}^{(j)}\\ &=(\sigma^{(j+1)})^{n+1}\partial_s^{2n+1}\kappa_t^{(j+1)}\lfloor_{at\,\vec{a}}+I_{2n+1}^{(j+1)}+\partial_t^n f^{(j)}\quad \textit{for}\ \ j\in\{1,2,3\},\end{aligned}$$

where $f^{(j)}$ *is the function defined by* (5.7)*, and* $I_{2n}^{(j)}$ *and* $I_{2n+1}^{(j)}$ *are represented by*

$$\begin{aligned}(6.15)\qquad I_{2n}^{(j)}&=n(\sigma^{(j)})^{n+1}\lambda_t^{(j)}\partial_s^{2n-1}\kappa_t^{(j)}\lfloor_{at\,\vec{a}}+Q(\vec{\sigma})R(\vec{\Theta})P_{2n+1}(\partial_s^{2n-2}\vec{\kappa}_t)\lfloor_{at\,\vec{a}}\\ &\quad+\sum_{m=1}^{n}\hat{Q}_m(\vec{\sigma},\partial_t^m\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+1-2m}(\partial_s^{2n-2m}\vec{\kappa}_t)\lfloor_{at\,\vec{a}},\end{aligned}$$

$$(6.16)\qquad I_{2n+1}^{(j)}=Q(\vec{\sigma})R(\vec{\Theta})P_{2n+2}(\partial_s^{2n}\vec{\kappa}_t)\lfloor_{at\,\vec{a}}+\sum_{m=1}^{n}\hat{Q}_m(\vec{\sigma},\partial_t^m\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+2-2m}(\partial_s^{2n+1-2m}\vec{\kappa}_t)\lfloor_{at\,\vec{a}}.$$

Furthermore, $\partial_t^n f^{(j)}$ *is a finite sum of*

$$\frac{Q_{n+1}(\vec{\sigma},\partial_t^{n+1}\Delta\vec{\alpha})}{(1-\langle\tau_t^{(j)},\tau_t^{(j+1)}\rangle)^{(2m_1+1)/2}(\sigma^{(j)}\sigma^{(j+1)})^{m_2}}\quad \textit{with}\ \ m_1,m_2\in\mathbb{N}.$$

Proof. We first prove the identity (6.13). Taking the time derivative of (2.6) and using (6.5), we have

$$0=\sum_{j=1}^{3}\Big(2\sigma^{(j)}\partial_\alpha\sigma^{(j)}(\partial_t\Delta^{(j)}\alpha)\kappa_t^{(j)}+(\sigma^{(j)})^3(\partial_s^2\kappa_t^{(j)}+P_3(\kappa_t^{(j)}))+(\sigma^{(j)})^2\lambda_t^{(j)}\partial_s\kappa_t^{(j)}\Big)\Big\lfloor_{\text{at }\vec{a}},$$

which implies that (6.13) with $n=1$ holds. Assume that (6.13) holds for some $n\in\mathbb{N}$. We then take the time derivative of (6.13) and apply (6.3), (6.4) and (6.5) to obtain

$$\begin{aligned}0=&\sum_{j=1}^{3}\partial_t((\sigma^{(j)})^{n+2}\partial_s^{2n}\kappa_t^{(j)})\lfloor_{\text{at }\vec{a}}+\partial_t(n(\sigma^{(j)})^{n+1}\lambda_t^{(j)}\partial_s^{2n-1}\kappa_t^{(j)})\lfloor_{\text{at }\vec{a}}\\ &\quad+\partial_t(Q(\vec{\sigma})P_{2n+1}(\partial_s^{2n-2}\kappa_t^{(j)}))\lfloor_{\text{at }\vec{a}}+\sum_{m=1}^{n}\partial_t(\hat{Q}_m(\vec{\sigma},\partial_t^m\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+1-2m}(\partial_s^{2n-2m}\vec{\kappa}_t))\lfloor_{\text{at }\vec{a}}\\ =&\sum_{j=1}^{3}\Big\{\hat{Q}_1(\vec{\sigma},\partial_t\Delta\vec{\alpha})\partial_s^{2n}\kappa_t^{(j)}+(\sigma^{(j)})^{n+3}\partial_s^{2n+2}\kappa_t^{(j)}+(\sigma^{(j)})^{n+2}\lambda_t^{(j)}\partial_s^{2n+1}\kappa_t^{(j)}+Q(\vec{\sigma})P_{2n+3}(\partial_s^{2n}\vec{\kappa}_t)\\ &\quad+\hat{Q}_1(\vec{\sigma},\partial_t\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+1}(\partial_s^{2n-1}\vec{\kappa}_t)+Q(\vec{\sigma})R(\vec{\Theta})P_{2n+3}(\partial_s^{2n}\vec{\kappa}_t)+n(\sigma^{(j)})^{n+2}\lambda_t^{(j)}\partial_s^{2n+1}\kappa_t^{(j)}\\ &\quad+\hat{Q}_1(\vec{\sigma},\partial_t\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+1}(\partial_s^{2n-2}\vec{\kappa}_t)+Q(\vec{\sigma})R(\vec{\Theta})P_{2n+3}(\partial_s^{2n}\vec{\kappa}_t)\\ &\quad+\sum_{m=1}^{n}\Big(\hat{Q}_{m+1}(\vec{\sigma},\partial_t^{m+1}\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+1-2m}(\partial_s^{2n-2m}\vec{\kappa}_t)\\ &\qquad+\hat{Q}_m(\vec{\sigma},\partial_t^m\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+3-2m}(\partial_s^{2n+2-2m}\vec{\kappa}_t)\Big)\Big\}\Big\lfloor_{\text{at }\vec{a}}\\ =&\sum_{j=1}^{3}\Big\{(\sigma^{(j)})^{n+3}\partial_s^{2n+2}\kappa_t^{(j)}+(n+1)(\sigma^{(j)})^{n+2}\lambda_t^{(j)}\partial_s^{2n+1}\kappa_t^{(j)}+Q(\vec{\sigma})R(\vec{\Theta})P_{2n+3}(\partial_s^{2n}\vec{\kappa}_t)\\ &\quad+\sum_{m=1}^{n+1}\hat{Q}_m(\vec{\sigma},\partial_t^m\Delta\vec{\alpha})R(\vec{\Theta})P_{2n+3-2m}(\partial_s^{2n+2-2m}\vec{\kappa}_t)\Big\}\Big\lfloor_{\text{at }\vec{a}},\end{aligned}$$

which coincides with (6.13) replaced n by $n+1$. We thus obtain (6.13) for any $n\in\mathbb{N}$.

The identity (6.14) follows from the time derivatives of (5.3) applying a similar calculation to obtain (6.13). The form of $\partial_t^n f^{(j)}$ can be obtained easily since $f^{(j)}$ is of the form

$$f^{(j)} = \frac{\hat{Q}_1(\vec{\sigma}, \partial_t \Delta\vec{\alpha})}{(1 - \langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle)^{1/2} (\sigma^{(j)}\sigma^{(j+1)})^2},$$

the formula $\langle \tau_t^{(j)}, \tau_t^{(j+1)} \rangle$ is given by (4.5) and the calculus rule (6.2) holds. □

Remark 6.9. Applying (6.10) and the boundedness of $L^{(j)}$ and $\sigma^{(j)}$, for any constants $A_l^{(j)}$ satisfying $\sum_{j=1}^3 \sum_{l=0}^{n-1} (l+1) A_l^{(j)} = n$, we obtain

$$\prod_{j=1}^{3} \left(\prod_{l=0}^{n-1} (\partial_t^l L^{(j)})^{A_l^{(j)}} \right) Q(\vec{\sigma})$$
$$\leq \begin{cases} \text{constant} & \text{if } A_l^{(j)} = 0 \text{ for any } l = 1, 2, \cdots, n-1, \quad j = 1, 2, 3, \\ \prod_{j=1}^3 \prod_{l=1}^{n-1} \left(P_{\leq 2l}(\|\partial_s^{2l-2} \vec{\kappa}_t\|) \right)^{A_l^{(j)}} & \text{otherwise.} \end{cases}$$

Since the inequalities

$$\prod_{j=1}^{3} \prod_{l=1}^{n-1} \left(P_{\leq 2l}(\|\partial_s^{2l-2} \vec{\kappa}_t\|) \right)^{A_l^{(j)}} \leq P_{\leq \sum_j \sum_l 2l A_l^{(j)}}(\|\partial_s^{2n-4} \vec{\kappa}_t\|)$$

and

$$\sum_{j=1}^{3} \sum_{l=1}^{n-1} 2l A_l^{(j)} \leq 2 \sum_{j=1}^{3} \sum_{l=0}^{n-1} (l+1) A_l^{(j)} \leq 2n$$

hold for $n \geq 2$, Lemma 6.7 implies

$$|\partial_t^n \Delta\alpha^{(j)}| \leq \begin{cases} M & \text{if } n = 1, \\ M + P_{\leq 2n}(\|\partial_s^{2n-4} \vec{\kappa}_t\|) & \text{if } n \geq 2 \end{cases} \tag{6.17}$$

for some constant M. It further implies

$$|\hat{Q}_{l_1}(\vec{\sigma}, \partial_t^{i_1} \Delta\vec{\alpha}) P_{l_2}(\partial_s^{i_2} \vec{\kappa}_t)| \leq P_{\leq 2l_1 + l_2}(\|\partial_s^{\max\{2i_1 - 4, i_2\}} \vec{\kappa}_t\|), \tag{6.18}$$

which will be used to estimate terms related to $I_l^{(j)}$ in Lemma 6.8. Further applying (4.20), we can obtain

$$|\partial_t^n f^{(j)}| \leq \begin{cases} M & \text{if } n = 1, \\ M + P_{\leq 2n+2}(\|\partial_s^{2n-2} \vec{\kappa}_t\|) & \text{if } n \geq 2 \end{cases} \tag{6.19}$$

for some constant M and any $j \in \{1, 2, 3\}$.

In section 5.2, we applied Lemma 5.12 to control the terms on the right hand side of (5.10) except the negative term. In this section, we will apply the following Gagliardo-Nirenberg interpolation inequalities instead of Lemma 5.12 to the higher order estimates. We refer to [1, 2] for the details of the interpolation inequalities.

Proposition 6.10. *Let Γ be a smooth plane curve with finite length L. If u is a smooth function defined on Γ, $m \geq 1$, $p \in [2, \infty]$ and $n \in \{0, 1, \cdots, m-1\}$, then there exist constants C_1 and C_2, which are independent of Γ, such that*

$$\|\partial_s^n u\|_{L^p} \leq C_1 \|\partial_s^m u\|_{L^2}^{\rho} \|u\|_{L^2}^{1-\rho} + \frac{C_2}{L^{m\rho}} \|u\|_{L^2}, \tag{6.20}$$

where

$$\rho = \begin{cases} \frac{n+1/2-1/p}{m} & \text{if} \quad p \in [2,\infty), \\ \frac{n+1/2}{m} & \text{if} \quad p = \infty. \end{cases}$$

We further derive estimates to apply to the polynomials P_l in the higher order estimates of the curvatures.

Lemma 6.11. *Let Γ be a smooth plane curve with finite length L. Let also $n \in \mathbb{N} \cup \{0\}$. Then, the following estimates hold.*

(i) If κ is a smooth function defined on Γ and non-negative integers A_l for $l = 0, 1, \cdots, n$ satisfy

$$\sum_{l=0}^{n} A_l \geq 2, \quad \sum_{l=0}^{n} (l+1) A_l \leq 2n+4.$$

Then, for any $\varepsilon > 0$, there exist constants $C_3 > 0$ and $q_1 > 0$ such that

$$\int_\Gamma \prod_{l=0}^{n} |\partial_s^l \kappa|^{A_l} \leq \varepsilon \int_\Gamma |\partial_s^{n+1}\kappa|^2 + |\kappa|^2 \, ds + C_3 \left(\int_\Gamma |\kappa|^2 \, ds \right)^{q_1}. \tag{6.21}$$

(ii) If κ is a smooth function defined on Γ and non-negative real values A_l for $l = 0, 1, \cdots, n$ satisfy

$$\sum_{l=0}^{n} A_l > 0, \quad \sum_{l=0}^{n} (l+1) A_l \leq 2n+3.$$

Then, for any $\varepsilon > 0$, there exist constants $C_4 > 0$ and $q_2 > 0$ such that

$$\prod_{l=0}^{n} \|\partial_s^l \kappa\|_{L^\infty}^{A_l} \leq \varepsilon \int_\Gamma |\partial_s^{n+1}\kappa|^2 + |\kappa|^2 \, ds + C_4 \left(\int_\Gamma |\kappa|^2 \, ds \right)^{q_2}. \tag{6.22}$$

(iii) If $\kappa^{(j)}$ is a smooth function defined on Γ for $j = 1, 2, 3$, $n \geq 1$ and non-negative real values $A_l^{(k)}$ for $l = 0, 1, \cdots, 2n$ and $k = 1, 2, 3$ satisfy

$$\sum_{k=1}^{3} \sum_{l=0}^{2n-2} (l+1) A_l^{(k)} \leq 2n.$$

Then, for any $\varepsilon > 0$ and $t > 0$, there exist constants $C_5, q_3 > 0$ and $q_4 > -1$ such that

$$\begin{aligned} & t^{2n-1} \left\| |\partial_s^{2n} \kappa^{(j)}| \cdot \prod_{k=1}^{3} \prod_{l=0}^{2n-2} |\partial_s^l \kappa^{(k)}|^{A_l^{(k)}} \right\|_{L^\infty} \\ \leq & \sum_{k=1}^{3} \varepsilon t^{2n} \int_\Gamma |\partial_s^{2n+1} \kappa^{(k)}|^2 + |\kappa^{(k)}|^2 \, ds + C_5 t^{q_4} \left(\int_\Gamma |\kappa^{(k)}|^2 \, ds \right)^{q_3} \end{aligned} \tag{6.23}$$

for any $j = 1, 2, 3$.

Proof. We first discuss the case (i). By the Hölder inequality, we have

$$\int_\Gamma \prod_{l=0}^{n} |\partial_s^l \kappa|^{A_l} \, ds \leq \prod_{l=0}^{n} \left(\int_\Gamma |\partial_s^l \kappa|^{A_l B_l} \right)^{1/B_l} = \prod_{l=0}^{n} \|\partial_s^l \kappa\|_{L^{A_l B_l}}^{A_l},$$

where the exponents B_l satisfy $\sum 1/B_l = 1$ and $A_l B_l \geq 2$ for every $l \in \{0, \cdots, n\}$ satisfying $A_l \neq 0$. Indeed, letting

$$B_l := \begin{cases} 1 & \text{if } \#\{l : A_l \neq 0\} = 1, \quad A_l \neq 0, \\ m & \text{if } \#\{l : A_l \neq 0\} = m, \quad A_l \neq 0, \end{cases}$$

where $\#\{l : A_l \neq 0\}$ is the number of non-zero exponents A_l, we can easily see that the exponents B_l satisfy the conditions by applying $\sum_{l=0}^n A_l \geq 2$. Therefore, due to (6.20), we have

$$\begin{aligned} \int_\Gamma \prod_{l=0}^n |\partial_s^l \kappa|^{A_l} \, ds &\leq C \prod_{l=0}^n \left(\|\partial_s^{n+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\rho_l A_l} \|\kappa\|_{L^2}^{(1-\rho_l)A_l} \\ &\leq C \left(\|\partial_s^{n+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\sum_{l=0}^n \rho_l A_l} \|\kappa\|_{L^2}^{\sum_{l=0}^n (1-\rho_l)A_l}, \end{aligned} \tag{6.24}$$

where $\rho_l = \frac{l+1/2-1/(A_l B_l)}{n+1}$ and C is some constant. Notice that $\sum_{l=0}^n A_l > 2$ or $\sum_{l=0}^n (l+1)A_l < 2n+4$ holds under the assumptions $2 \leq \sum_{l=0}^n A_l$ and $\sum_{l=0}^n (l+1)A_l \leq 2n+4$. We then have

$$\sum_{l=0}^n \rho_l A_l = \sum_{l=0}^n \frac{(l+1/2)A_l - 1/B_l}{n+1} = \frac{\sum_{l=0}^n (l+1)A_l - \sum_{l=0}^n A_l/2 - 1}{n+1} < 2.$$

Therefore, we can proceed to estimate (6.24) by applying Young's inequality as

$$\int_\Gamma \prod_{l=0}^n |\partial_s^l \kappa|^{A_l} \, ds \leq \varepsilon \int_\Gamma |\partial_s^{n+1}\kappa|^2 + |\kappa|^2 \, ds + C_3 \left(\int_\Gamma |\kappa|^2 \, ds \right)^{\frac{2\sum_{l=0}^n (1-\rho_l)A_l}{2-\sum_{l=0}^n \rho_l A_l}}$$

and also the positivity of the exponent $(2\sum(1-\rho_l)A_l)/(2-\sum \rho_l A_l)$ can be seen. Letting $q_1 = (2\sum(1-\rho_l)A_l)/(2-\sum \rho_l A_l)$, we have (6.21).

For the case (ii), we have by applying (6.20)

$$\begin{aligned} \left\| \prod_{l=0}^n |\partial_s^l \kappa|^{A_l} \right\|_{L^\infty} = \prod_{l=0}^n \|\partial_s^l \kappa\|_{L^\infty}^{A_l} &\leq C \prod_{l=0}^n \left(\|\partial_s^{n+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\rho_l A_l} \|\kappa\|_{L^2}^{(1-\rho_l)A_l} \\ &\leq C \left(\|\partial_s^{n+1}\kappa\|_{L^2} + \|\kappa\|_{L^2} \right)^{\sum_{l=0}^n \rho_l A_l} \|\kappa\|_{L^2}^{\sum_{l=0}^n (1-\rho_l)A_l}, \end{aligned}$$

where $\rho_l = \frac{l+1/2}{n+1}$ and C is some constant. Since $\sum A_l \geq \sum (l+1)A_l/(n+1)$, we have by $\sum(l+1)A_l \leq 2n+3$

$$\begin{aligned} \sum_{l=0}^n \rho_l A_l &= \frac{\sum_{l=0}^n (l+1)A_l - \frac{1}{2}\sum_{l=0}^n A_l}{n+1} \leq \frac{\sum_{l=0}^n (l+1)A_l - \frac{1}{2}\sum_{l=0}^n (l+1)A_l/(n+1)}{n+1} \\ &= \frac{(2n+1)\sum_{l=0}^n (l+1)A_l}{2(n+1)^2} \leq \frac{(2n+1)(2n+3)}{2(n+1)^2} \leq 2 - \frac{1}{2(n+1)^2} < 2. \end{aligned}$$

Therefore, we can obtain (6.22) by Young's inequality as in the case (i).

The estimate in the case (iii) also can be proved by a similar argument. Due to the calculations in the case (ii), we have

$$\begin{aligned} &t^{2n-1} \left\| |\partial_s^{2n}\kappa^{(j)}| \cdot \prod_{k=1}^3 \prod_{l=0}^{2n-2} |\partial_s^l \kappa^{(k)}|^{A_l^{(k)}} \right\|_{L^\infty} \\ &\leq C (t^n \|\partial_s^{2n+1}\kappa^{(j)}\|_{L^2} + t^n \|\kappa^{(j)}\|_{L^2})^\rho \left(\prod_{k=1}^3 \left(t^n \|\partial_s^{2n+1}\kappa^{(k)}\|_{L^2} + t^n \|\kappa^{(k)}\|_{L^2} \right)^{\sum_{l=0}^{2n-2} \rho_l^{(k)} A_l^{(k)}} \right). \end{aligned}$$

$$t^{2n-1-n(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)})}\|\kappa^{(j)}\|_{L^2}^{1-\rho}\cdot\prod_{k=1}^{3}\|\kappa^{(k)}\|_{L^2}^{\sum_{l=0}^{2n-2}(1-\rho_l^{(k)})A_l^{(k)}},$$

where $\rho=\frac{2n+1/2}{2n+1}$, $\rho_l^{(k)}=\frac{l+1/2}{2n+1}$ and C is some constant. Since the geometric order of the left hand side is not larger than $4n+1$, which is smaller than twice of the highest order of the derivative in right hand side plus 3, we can see

$$\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)}<2$$

as in the case (ii). Therefore, applying Young's inequality, we have, for some $C_5, q_3>0$,

$$t^{2n-1}\left\||\partial_s^{2n}\kappa^{(j)}|\cdot\prod_{k=1}^{3}\prod_{l=0}^{2n-2}|\partial_s^l\kappa^{(k)}|^{A_l^{(k)}}\right\|_{L^\infty}\le\sum_{k=1}^{3}\varepsilon t^{2n}\int_\Gamma|\partial_s^{2n+1}\kappa^{(k)}|^2+|\kappa^{(k)}|^2\,ds+C_5t^{q_4}\left(\int_\Gamma|\kappa^{(k)}|^2\,ds\right)^{q_3}$$

with

$$q_4=\frac{2(2n-1-n(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)}))}{2-(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)})}.$$

Since $q_4>-1$ is equivalent to

$$\begin{aligned}0&<2\left(2n-1-n\left(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)}\right)\right)+2-\left(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)}\right)\\&=4n-(2n+1)\left(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)}\right),\end{aligned}$$

we prove the positivity to obtain $q_4>-1$. Due to $\sum_l A_l^{(k)}\ge\sum_l(l+1)A_l/(2n-1)$ and $\sum_k\sum_l(l+1)A_l^{(k)}\le 2n$, we have

$$\begin{aligned}&4n-(2n+1)\left(\rho+\sum_{k=1}^{3}\sum_{l=0}^{2n-2}\rho_l^{(k)}A_l^{(k)}\right)\\&=2n-\frac{1}{2}-\sum_{k=1}^{3}\sum_{l=1}^{2n-2}(l+1)A_l^{(k)}+\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{2n-2}A_l^{(k)}\\&\ge 2n-\frac{1}{2}-\sum_{k=1}^{3}\sum_{l=1}^{2n-2}(l+1)A_l^{(k)}+\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{2n-2}\frac{(l+1)A_l^{(k)}}{2n-1}\\&=2n-\frac{1}{2}-\frac{4n-3}{4n-2}\sum_{k=1}^{3}\sum_{l=1}^{2n-2}(l+1)A_l^{(k)}\ge 2n-\frac{1}{2}-\frac{4n-3}{4n-2}\cdot 2n=\frac{1}{4n-2}>0.\end{aligned}$$

We thus obtain the conclusion. □

We now prove the higher order estimate of the curvatures for any $k\in\mathbb{N}$ applying the exponential L^2-decay of the curvatures. Note that the weights t^m in the following proposition enable us to obtain not only the decay estimate but also the smoothing effect, namely, the following estimate is independent of the derivatives of the initial curvatures.

Proposition 6.12. *Assume (A1)–(A3). Let a pair of* $\{\Gamma_t^{(j)}\}_{j\in\{1,2,3\}}$ *and* $\vec{\alpha}$ *be a smooth geometric flow governed by* (1.3)–(1.7) *in the time interval* $[0,T)$. *Let also* m_3 *be the constant in Lemma 4.2. Then, for any* $n \in \mathbb{N}$, *there exist* $\varepsilon_5, c_{13}, c_{14}, c_{15}, c_{16} > 0$ *and* $c_{17} > -1$ *such that if*

$$E(0) \le \sigma(0)m_3, \quad \sum_{j=1}^{3}\left(\Delta^{(j)}\alpha_0\right)^2 \le \varepsilon_5, \quad \sum_{j=1}^{3}\int_{\Gamma_0^{(j)}}\left(\sigma(\Delta^{(j)}\alpha_0)\right)^2\left(\kappa_0^{(j)}\right)^2 ds \le \varepsilon_5, \tag{6.25}$$

then

$$\begin{aligned}&\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}\frac{t^m}{m!}(\sigma^{(j)})^{m+2}(\partial_s^m\kappa_t^{(j)})^2\,ds\\ &\le c_{13}\int_0^t\left(\sum_{m=0}^{2n}\tilde{t}^m+\tilde{t}^{c_{17}}\right)e^{-c_{14}\tilde{t}}\,d\tilde{t}+c_{15}\sum_{m'=1}^{n}t^{m'}e^{-c_{16}t}+2\sum_{j=1}^{3}\int_{\Gamma_0^{(j)}}(\sigma_0^{(j)})^2(\kappa_0^{(j)})^2\,ds\end{aligned}\tag{6.26}$$

for any $t \in [0,T)$.

Proof. Note that the length $L^{(j)}$ is uniformly positive and bounded for $j \in \{1,2,3\}$ due to Lemma 4.1, Lemma 4.2 and the first assumption in (6.25). The uniformly boundedness of $|\partial_\alpha^m\sigma^{(j)}|$ for $m \in \mathbb{N}\cup\{0\}$ and $j \in \{1,2,3\}$ also follows from (4.9) and (6.25). We will choose ε_5 in (6.25) small so that Corollary 4.7, Corollary 4.10 and Proposition 5.13 can be applied. In this setting, we can regard the functions R and Q as bounded coefficients, as we remarked in Remark 6.2, and this boundedness will be used through this proof without citing. Note also that we can use the estimates (6.18) and (6.19) in this setting. We first derive a higher order energy type identity. Due to (2.3), (2.5), (6.5) and (6.6), for any $j \in \{1,2,3\}$, we have by a similar calculation to (5.11)

$$\begin{aligned}&\frac{d}{dt}\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}\frac{t^m}{m!}(\sigma^{(j)})^{m+2}(\partial_s^m\kappa_t^{(j)})^2\,ds\\ =&-2\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}\frac{t^m}{m!}(\sigma^{(j)})^{m+3}(\partial_s^{m+1}\kappa_t^{(j)})^2\,ds+\int_{\Gamma_t^{(j)}}\sum_{m=1}^{2n}\frac{t^{m-1}}{(m-1)!}(\sigma^{(j)})^{m+2}(\partial_s^m\kappa_t^{(j)})^2\,ds\\ &+\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}t^m\left\{Q(\sigma^{(j)},\partial_t\Delta^{(j)}\alpha)P'_{2m+2}(\partial_s^m\kappa_t^{(j)})+Q(\sigma^{(j)},\partial_t\Delta^{(j)}\alpha)P'_{2m+4}(\partial_s^m\kappa_t^{(j)})\right\}ds\\ &+\sum_{m=0}^{2n}\frac{t^m}{m!}\left\{2(\sigma^{(j)})^{m+3}\partial_s^{m+1}\kappa_t^{(j)}\partial_s^m\kappa_t^{(j)}+(\sigma^{(j)})^{m+2}\lambda_t^{(j)}(\partial_s^m\kappa_t^{(j)})^2\right\}\Big\lfloor_{\text{at }\vec{a}},\end{aligned}\tag{6.27}$$

where $Q(\sigma^{(j)},\partial_t\Delta^{(j)}\alpha)$ is a polynomial (for each term) in $\sigma^{(j)}, \partial_\alpha\sigma^{(j)}$ and $\partial_t\Delta^{(j)}\alpha$, and thus Q is uniformly bounded. Due to (6.21) and $\sigma(0) \le \sigma^{(j)}$, the third integral on the right hand side of (6.27) can be divided by a higher order term and the L^2-norm of the curvature as

$$\begin{aligned}&\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}t^m\left\{Q(\sigma^{(j)},\partial_t\Delta^{(j)}\alpha)P'_{2m+2}(\partial_s^m\kappa_t^{(j)})+Q(\sigma^{(j)},\partial_t\Delta^{(j)}\alpha)P'_{2m+4}(\partial_s^m\kappa_t^{(j)})\right\}ds\\ \le&\frac{1}{2}\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}\frac{t^m}{m!}(\sigma^{(j)})^{m+3}(\partial_s^{m+1}\kappa_t^{(j)})^2\,ds+M\sum_{m=0}^{2m}t^m\|\kappa_t^{(j)}\|_{L^2}^{q'_m}\end{aligned}\tag{6.28}$$

for some positive constants M, q'_m with $m \in \{0, 1, \cdots, 2n\}$. Note that although we may obtain the terms $\|\kappa_t^{(j)}\|_{L^2}^{q'_m}$ with some different exponents q'_m, we can here replace the different exponents by a same exponent q'_m, due to the exponential L^2-decay of the curvature as in Proposition 5.13, by choosing q'_m small and M large depending on the weighted L^2-norm of the initial curvatures and the initial misorientations if necessary. For example, for any $0 < q'_{m,1} < q'_{m,2}$, we have

$$\|\kappa_t^{(j)}\|_{L^2}^{q'_{m,1}} + \|\kappa_t^{(j)}\|_{L^2}^{q'_{m,2}} = (1 + \|\kappa_t^{(j)}\|_{L^2}^{q'_{m,2}-q'_{m,1}})\|\kappa_t^{(j)}\|_{L^2}^{q'_{m,1}} \le M\|\kappa_t^{(j)}\|_{L^2}^{q'_{m,1}},$$

where M is a constant depending on the weighted L^2-norm of the initial curvatures and the initial misorientations. Since, on the right hand side of (6.27), the second integral can be absorbed into the first integral, we see that

$$\begin{aligned} &\frac{d}{dt}\int_{\Gamma_t^{(j)}} \sum_{m=0}^{2n} \frac{t^m}{m!} (\sigma^{(j)})^{m+2} (\partial_s^m \kappa_t^{(j)})^2 \, ds \\ \le\ & -\frac{1}{2}\int_{\Gamma_t^{(j)}} \sum_{m=0}^{2n} \frac{t^m}{m!} (\sigma^{(j)})^{m+3} (\partial_s^{m+1} \kappa_t^{(j)})^2 \, ds + M \sum_{m=0}^{2m} t^m \|\kappa_t^{(j)}\|_{L^2}^{q'_m} \\ & + \sum_{m=0}^{2n} \frac{t^m}{m!} \Big\{ 2(\sigma^{(j)})^{m+3} \partial_s^{m+1}\kappa_t^{(j)} \partial_s^m \kappa_t^{(j)} + (\sigma^{(j)})^{m+2} \lambda_t^{(j)} (\partial_s^m \kappa_t^{(j)})^2 \Big\} \Big\lfloor_{\text{at } \vec{a}} \end{aligned} \tag{6.29}$$

by applying the inequality (6.28). Therefore, we continue to estimate the boundary terms. We now let

$$J_m^{(j)} := \frac{t^m}{m!} \Big\{ 2(\sigma^{(j)})^{m+3} \partial_s^{m+1}\kappa_t^{(j)} \partial_s^m \kappa_t^{(j)} + (\sigma^{(j)})^{m+2} \lambda_t^{(j)} (\partial_s^m \kappa_t^{(j)})^2 \Big\} \Big\lfloor_{\text{at } \vec{a}}$$

for $m \in \{0, 1, \cdots, 2n\}$ and $j \in \{1, 2, 3\}$.

Hereafter we omit the trace operator $\lfloor_{\text{at } \vec{a}}$ since we discuss only the boundary terms at the junction point $\vec{a}$ for a while. The boundary term $J_0^{(j)}$ can be estimated as in (5.11), (5.12). We thus obtain, due to (2.6), (5.3), (5.13), (6.3) and (6.19),

$$\begin{aligned} \sum_{j=1}^3 J_0^{(j)} \le\ & \sum_{j=1}^3 |Q(\vec{\sigma})| P_1(\|\vec{\kappa}_t\|) |f^{(j)}| + |Q(\vec{\sigma})| \cdot \big|\lambda_t^{(j)}\big| P_2(\|\kappa_t^{(j)}\|) \\ \le\ & |Q(\vec{\sigma}) R(\vec{\Theta})| (P_1(\|\vec{\kappa}_t\|) + P_3(\|\vec{\kappa}_t\|)). \end{aligned} \tag{6.30}$$

This kind of strategy can be applied to $J_{2m'}^{(j)}$ for $m' \in \mathbb{N}$. Indeed, since (6.3), (6.15) and (6.16) show

$$\begin{aligned} &(\sigma^{(j)})^{2m'+2} \lambda_t^{(j)} (\partial_s^{2m'} \kappa_t^{(j)})^2 = Q(\vec{\sigma}) R(\vec{\Theta}) P_{4m'+3}(\partial_s^{2m'} \vec{\kappa}_t), \\ &(\sigma^{(j)})^{m'+2} \partial_s^{2m'} \kappa_t^{(j)} I_{2m'+1}^{(j)} = Q(\vec{\sigma}) R(\vec{\Theta}) P_{4m'+3}(\partial_s^{2m'} \vec{\kappa}_t) + \sum_{l=1}^{m'} \hat{Q}_l(\vec{\sigma}, \partial_t^l \Delta\vec{\alpha}) R(\vec{\Theta}) P_{4m'+3-2l}(\partial_s^{2m'} \vec{\kappa}_t), \\ &(\sigma^{(j)})^{m'+1} \partial_s^{2m'+1} \kappa_t^{(j)} I_{2m'}^{(j)} = m' (\sigma^{(j)})^{2m'+2} \lambda_t^{(j)} \partial_s^{2m'+1} \kappa_t^{(j)} \partial_s^{2m'-1} \kappa_t^{(j)} \\ &\qquad + \partial_s^{2m'+1} \kappa_t^{(j)} \Big(Q(\vec{\sigma}) R(\vec{\Theta}) P_{2m'+1}(\partial_s^{2m'-2} \vec{\kappa}_t) \\ &\qquad\qquad + \sum_{l=1}^{m'} \hat{Q}_l(\vec{\sigma}, \partial_t^l \Delta\vec{\alpha}) R(\vec{\Theta}) P_{2m'+1-2l}(\partial_s^{2m'-2l} \vec{\kappa}_t) \Big), \end{aligned}$$

$$I^{(j)}_{2m'}I^{(j)}_{2m'+1} = Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+3}(\partial_s^{2m'}\vec{\kappa}_t) + \sum_{l=1}^{2m'}\hat{Q}_l(\vec{\sigma},\partial_t^{\min\{m',l\}}\Delta\vec{\alpha})R(\vec{\Theta})P_{4m'+3-2l}(\partial_s^{2m'}\kappa_t^{(j)}),$$

we obtain

$$\begin{aligned}
J^{(j)}_{2m'} &= \frac{t^{2m'}}{(2m')!}\Big\{2((\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I^{(j)}_{2m'})((\sigma^{(j)})^{m'+1}\partial_s^{2m'+1}\kappa_t^{(j)} + I^{(j)}_{2m'+1}) - 2I^{(j)}_{2m'}I^{(j)}_{2m'+1} \\
&\quad - (\sigma^{(j)})^{2m'+2}\lambda_t^{(j)}(\partial_s^{2m'}\kappa_t^{(j)})^2 - 2(\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)}I^{(j)}_{2m'+1} - 2(\sigma^{(j)})^{m'+1}\partial_s^{2m'+1}\kappa_t^{(j)}I_{2m'}\Big\} \\
&= \frac{t^{2m'}}{(2m')!}2((\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I^{(j)}_{2m'})((\sigma^{(j)})^{m'+1}\partial_s^{2m'+1}\kappa_t^{(j)} + I^{(j)}_{2m'+1}) \\
&\quad - \frac{t^{2m'}}{(2m'-1)!}(\sigma^{(j)})^{2m'+2}\lambda_t^{(j)}\partial_s^{2m'+1}\kappa_t^{(j)}\partial_s^{2m'-1}\kappa_t^{(j)} \\
&\quad + t^{2m'}\partial_s^{2m'+1}\kappa_t^{(j)}\Big(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'+1}(\partial_s^{2m'-2}\vec{\kappa}_t) \\
&\qquad + \sum_{l=1}^{m'}\hat{Q}_l(\vec{\sigma},\partial_t^l\Delta\vec{\alpha})R(\vec{\Theta})P_{2m'+1-2l}(\partial_s^{2m'-2l}\vec{\kappa}_t)\Big) \\
&\quad + t^{2m'}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+3}(\partial_s^{2m'}\vec{\kappa}_t) + \sum_{l=1}^{2m'}\hat{Q}_{2l}(\vec{\sigma},\partial_t^{\min\{m',l\}}\Delta\vec{\alpha})R(\vec{\Theta})P_{4m'+3-2l}(\partial_s^{2m'}\kappa_t^{(j)})\right).
\end{aligned} \tag{6.31}$$

For the first term, we can apply (5.13) with

$$A^{(j)} = (\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I^{(j)}_{2m'}, \quad B^{(j)} = (\sigma^{(j)})^{m'+1}\partial_s^{2m'+1}\kappa_t^{(j)} + I^{(j)}_{2m'+1},$$

(6.13), (6.14) and (6.19) to obtain

$$\begin{aligned}
&\sum_{j=1}^{3}\frac{t^{2m'}}{(2m')!}2((\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I^{(j)}_{2m'})((\sigma^{(j)})^{m'+1}\partial_s^{2m'+1}\kappa_t^{(j)} + I^{(j)}_{2m'+1}) \\
&= \sum_{j=1}^{3}\frac{2t^{2m'}}{3\cdot(2m')!}\left((\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I^{(j)}_{2m'}\right)(\partial_t^{m'}f^{(j)} - \partial_t^{m'}f^{(j-1)}) \le t^{2m'}P_{\le 4m'+3}(\|\partial_s^{2m'}\vec{\kappa}_t\|).
\end{aligned}$$

Substituting it into the sum of (6.31) and using (6.18), we have

$$\begin{aligned}
\sum_{j=1}^{3}J^{(j)}_{2m'} \le \sum_{j=1}^{3}\Big\{ &-\frac{t^{2m'}}{(2m'-1)!}(\sigma^{(j)})^{2m'+2}\lambda_t^{(j)}\partial_s^{2m'+1}\kappa_t^{(j)}\partial_s^{2m'-1}\kappa_t^{(j)} \\
&+ t^{2m'}\partial_s^{2m'+1}\kappa_t^{(j)}\Big(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'+1}(\partial_s^{2m'-2}\vec{\kappa}_t) \\
&\quad + \sum_{l=1}^{m'}\hat{Q}_l(\vec{\sigma},\partial_t^l\Delta\vec{\alpha})R(\vec{\Theta})P_{2m'+1-2l}(\partial_s^{2m'-2l}\vec{\kappa}_t)\Big)\Big\} + t^{2m'}P_{\le 4m'+3}(\|\partial_s^{2m'}\vec{\kappa}_t\|).
\end{aligned} \tag{6.32}$$

We here obtained critical terms with the coefficient $\partial_s^{2m'+1}\kappa_t^{(j)}$, and thus we will change the terms to a time derivative of lower order terms using (6.5) and also (6.3), (6.4) as

$$-\frac{t^{2m'}}{(2m'-1)!}(\sigma^{(j)})^{2m'+2}\lambda_t^{(j)}\partial_s^{2m'+1}\kappa_t^{(j)}\partial_s^{2m'-1}\kappa_t^{(j)} + t^{2m'}\partial_s^{2m'+1}\kappa_t^{(j)}\Big(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'+1}(\partial_s^{2m'-2}\vec{\kappa}_t)\Big)$$

$$
\begin{aligned}
&= -\frac{t^{2m'}}{(2m'-1)!}(\sigma^{(j)})^{2m'+1}\lambda_t^{(j)}\partial_t\partial_s^{2m'-1}\kappa_t^{(j)}\partial_s^{2m'-1}\kappa_t^{(j)} \\
&\quad + t^{2m'}\partial_t\partial_s^{2m'-1}\kappa_t^{(j)}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'+1}(\partial_s^{2m'-2}\vec{\kappa}_t)\right) + t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+3}(\partial_s^{2m'}\vec{\kappa}_t) \\
&= \partial_t\left(-\frac{t^{2m'}}{2\cdot(2m'-1)!}(\sigma^{(j)})^{2m'+1}\lambda_t^{(j)}(\partial_s^{2m'-1}\kappa_t^{(j)})^2 + t^{2m'}\partial_s^{2m'-1}\kappa_t^{(j)}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'+1}(\partial_s^{2m'-2}\vec{\kappa}_t)\right)\right) \\
&\quad + t^{2m'-1}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t) + t^{2m'}\hat{Q}_1(\vec{\sigma},\partial_t\Delta\vec{\alpha})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t) \\
&\quad + t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+3}(\partial_s^{2m'}\vec{\kappa}_t) \\
&= \partial_t\left(t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t)\right) + t^{2m'-1}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t) \\
&\quad + t^{2m'}\hat{Q}_1(\vec{\sigma},\partial_t\Delta\vec{\alpha})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t) + t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+3}(\partial_s^{2m'}\vec{\kappa}_t),
\end{aligned}
$$

which implies, estimating the non-differential terms as in (6.32) and substituting it to (6.32),

$$
\sum_{j=1}^{3} J_{2m'}^{(j)} \le \partial_t\left(t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t)\right) + t^{2m'-1}P_{4m'+1}(\|\partial_s^{2m'-1}\vec{\kappa}_t\|) + t^{2m'}P_{\le 4m'+3}(\|\partial_s^{2m'}\vec{\kappa}_t\|).
$$

Since Young's inequality implies

$$
P_{\le l}(\|\partial_s^i\vec{\kappa}_t\|) \le \sum_{j=1}^{3} P_{\le l}(\|\partial_s^i\kappa_t^{(j)}\|),
$$

the estimate (6.22) can be applied to the non-differential terms as

$$
\begin{aligned}
\sum_{j=1}^{3} J_{2m'}^{(j)} &\le \partial_t\left(t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t)\right) \\
&\quad + \sum_{j=1}^{3}\frac{1}{8}\int_{\Gamma_t^{(j)}}\frac{t^{2m'-1}}{(2m'-1)!}(\sigma^{(j)})^{2m'+2}(\partial_s^{2m'}\kappa_t^{(j)})^2 + \frac{t^{2m'}}{(2m')!}(\sigma^{(j)})^{2m'+3}(\partial_s^{2m'+1}\kappa_t^{(j)})^2\,ds \\
&\quad + \tilde{M}\sum_{j=1}^{3}(t^{2m'-1}\|\kappa_t^{(j)}\|_{L^2}^{\tilde{q}'_{2m'-1}} + t^{2m'}\|\kappa_t^{(j)}\|_{L^2}^{\tilde{q}'_{2m'}})
\end{aligned} \tag{6.33}
$$

for some positive constants $\tilde{M}, \tilde{q}'_{2m'-1}$ and $\tilde{q}'_{2m'}$. The strategy for critical terms in the estimate of $J_{2m'}^{(j)}$ is not necessary to $J_{2m'-1}^{(j)}$ since the estimate (6.23) can be applied to critical terms in the following estimate of $J_{2m'-1}^{(j)}$. Indeed, as the calculation in (6.31), applying (6.3), (6.15) and (6.16), we have

$$
\begin{aligned}
J_{2m'-1}^{(j)} &= 2\frac{t^{2m'-1}}{(2m'-1)!}((\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I_{2m'}^{(j)})((\sigma^{(j)})^{m'}\partial_s^{2m'-1}\kappa_t^{(j)} + I_{2m'-1}^{(j)}) \\
&\quad + t^{2m'-1}\partial_s^{2m'}\kappa_t^{(j)}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'}(\partial_s^{2m'-2}\vec{\kappa}_t) + \sum_{l=1}^{m'-1}\hat{Q}_l(\vec{\sigma},\partial_t^l\Delta\vec{\alpha})R(\vec{\Theta})P_{2m'-2l}(\partial_s^{2m'-1-2l}\vec{\kappa}_t)\right) \\
&\quad + t^{2m'-1}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t) + \sum_{l=1}^{2m'-1}\hat{Q}_l(\vec{\sigma},\partial_t^{\min\{m',l\}}\Delta\vec{\alpha})R(\vec{\Theta})P_{4m'+1-2l}(\partial_s^{2m'-1}\vec{\kappa}_t)\right),
\end{aligned}
$$

which implies, applying the formulas (5.13), (6.13) and (6.14) to the first term and the estimates (6.18) and (6.19) to $\hat{P}_l$ and $\partial_t^{m'-2} f^{(j)}$, respectively,

$$\begin{aligned}\sum_{j=1}^{3} J_{2m'-1}^{(j)} = & \sum_{j=1}^{3} \frac{2t^{2m'-1}}{3\cdot(2m'-1)!}\left((\sigma^{(j)})^{m'+2}\partial_s^{2m'}\kappa_t^{(j)} + I_{2m'}^{(j)}\right)\kappa_t^{(j)}(\partial_t^{m'-1}f^{(j)} - \partial_t^{m'-1}f^{(j-1)}) \\ & + t^{2m'-1}\partial_s^{2m'}\kappa_t^{(j)}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{2m'}(\partial_s^{2m'-2}\vec{\kappa}_t) + \sum_{l=1}^{m'-1}\hat{Q}_l(\vec{\sigma},\partial_t^l\Delta\vec{\alpha})R(\vec{\Theta})P_{2m'-2l}(\partial_s^{2m'-1-2l}\vec{\kappa}_t)\right) \\ & + t^{2m'-1}\left(Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t) + \sum_{l=1}^{2m'-1}\hat{Q}_l(\vec{\sigma},\partial_t^{\min\{m',l\}}\Delta\vec{\alpha})R(\vec{\Theta})P_{4m'+1-2l}(\partial_s^{2m'-1}\vec{\kappa}_t)\right) \\ \le & \sum_{j=1}^{3} t^{2m'-1}|\partial_s^{2m'}\kappa_t^{(j)}|(P_{\le 2m'}(\|\partial_s^{2m'-2}\vec{\kappa}_t\|) + \hat{M}') + t^{2m'-1}P_{\le 4m'+1}(\|\partial_s^{2m'-1}\vec{\kappa}_t\|)\end{aligned}$$

for some positive constant $\hat{M}'$. Applying (6.23) to the critical terms with the coefficient $\partial_s^{2m'}\kappa_t^{(j)}$ and (6.22) to the remained terms, we have

$$\begin{aligned}\sum_{j=1}^{3} J_{2m'-1}^{(j)} \le & \sum_{j=1}^{3}\frac{1}{8}\int_{\Gamma_t^{(j)}}\frac{t^{2m'-1}}{(2m'-1)!}(\sigma^{(j)})^{2m'+2}(\partial_s^{2m'}\kappa_t^{(j)})^2 + \frac{t^{2m'}}{(2m')!}(\sigma^{(j)})^{2m'+3}(\partial_s^{2m'+1}\kappa_t^{(j)})^2\,ds \\ & + \hat{M}\sum_{j=1}^{3}(t^{2m'-1}\|\kappa_t^{(j)}\|_{L^2}^{\hat{q}_{2m'-1}} + t^{r_{2m'-1}}\|\kappa_t^{(j)}\|_{L^2}^{\hat{q}'_{2m'-1}})\end{aligned} \tag{6.34}$$

for some constants $\hat{M}, \hat{q}_{2m'-1}, \hat{q}'_{2m'-1} > 0$ and $r_{2m'-1} > -1$.

We now substitute (6.30), (6.33) and (6.34) into the summation of (6.29) with respect to $j \in \{1,2,3\}$ and apply the exponential L^2-decay of the curvatures as in Proposition 5.13 to obtain

$$\begin{aligned} & \frac{d}{dt}\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}\frac{t^m}{m!}(\sigma^{(j)})^{m+2}(\partial_s^m\kappa_t^{(j)})^2\,ds \\ \le & \sum_{m'=1}^{n}\partial_t\left(t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t)\lfloor_{\text{at }\vec{a}}\right) + \frac{c_{13}}{2}\left(\sum_{m=0}^{2n}t^m + t^{c_{17}}\right)e^{-c_{14}t}\end{aligned}$$

for some constants $c_{13}, c_{14} > 0$ and $c_{17} > -1$. Integrating it and applying (6.22) again to the polynomials $P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t)\lfloor_{\text{at }\vec{a}}$, we have by means of the L^2-exponential decay of the curvatures as in Proposition 5.13

$$\begin{aligned} & \sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\sum_{m=0}^{2n}\frac{t^m}{m!}(\sigma^{(j)})^{m+2}(\partial_s^m\kappa_t^{(j)})^2\,ds - \sum_{j=1}^{3}\int_{\Gamma_0^{(j)}}(\sigma_0^{(j)})^2(\kappa_0^{(j)})^2\,ds \\ \le & \sum_{m'=1}^{n}t^{2m'}Q(\vec{\sigma})R(\vec{\Theta})P_{4m'+1}(\partial_s^{2m'-1}\vec{\kappa}_t)\lfloor_{\text{at }\vec{a}} + \frac{c_{13}}{2}\int_0^t\left(\sum_{m=0}^{2n}\tilde{t}^m + \tilde{t}^{c_{17}}\right)e^{-c_{14}\tilde{t}}\,d\tilde{t} \\ \le & \frac{1}{2}\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}\sum_{m'=1}^{n}\frac{t^{2m'}}{(2m')!}(\sigma^{(j)})^{2m'+2}(\partial_s^{2m'}\kappa_t^{(j)})^2\,ds + \frac{c_{13}}{2}\int_0^t\left(\sum_{m=0}^{2n}\tilde{t}^m + \tilde{t}^{c_{17}}\right)e^{-c_{14}\tilde{t}}\,d\tilde{t} + \frac{c_{15}}{2}\sum_{m'=1}^{n}t^{2m'}e^{-c_{16}t}\end{aligned}$$

for some $c_{15}, c_{16} > 0$, which implies (6.26). □

Remark 6.13. (i) The Rayleigh quotient in Section 5.1 cannot be applied to the negative terms in (6.29) because of the difference of the boundary conditions for the curvature and those derivatives. Therefore, we cannot obtain exponential decay of the higher order derivatives of the curvatures directly extending the method for the exponential L^2-decay of the curvatures.

(ii) In Proposition 6.12, we assumed (A1)–(A3) and the smallness of the initial datum. On the other hand, under only the assumptions (A1) and (A3), the boundedness of $\sup_{j\in\{1,2,3\},\, t\in[t_0,T]} \|\kappa_t^{(j)}\|_{H^k}$ for each $t_0 \in (0,T)$ and $k \in \mathbb{N}$ can be proved if the following properties hold for $t \leq T$ and $j \in \{1,2,3\}$;

(a) $\|\kappa_t^{(j)}\|_{L^2}$ is bounded;
(b) $L^{(j)}(t)$ is positive;
(c) $\Theta^{(j+1)} - \Theta^{(j)} \in (0,\pi)$ at the junction point $\vec{a}$ or (4.6) holds.

This kind of H^k-boundedness was proved in [34] for the classical curvature flow applying energy type estimates as in the proof of Proposition 6.12, and thus we refer to [34] for the detailed proof. We here remark that some estimates of the misorientation parameters are needed in addition to the argument in [34]. As we discussed in Remark 4.8, the boundedness of $\sigma(\Delta^{(j)}\alpha)$ and $\partial_t\Delta^{(j)}\alpha$ can be obtained only assuming (A1) and (A3). Therefore, according to the discussions in Remark 6.6 and Remark 6.9, the inequalities (6.10), (6.17) and (6.18) can be obtained assuming only (A1) and (A3). We also remark that the estimate to control f as in (6.19) can be derived assuming (A1), (A3) and the property (c) above. By means of these estimates, we can extend the argument as in [34] to obtain the H^2-boundedness of the curvatures in our problem without assuming (A2) and the smallness of the initial datum.

While we already derived the higher order estimates of the curvatures, C^∞ or the Hölder estimates of the angle function $\Theta^{(j)}$ defined by (2.1) are not obvious and are necessary to discuss the maximum existence time via the existence theorem as in Proposition 2.14. Therefore, we give the following corollary.

Corollary 6.14. *Let $\Theta^{(j)}$ be the angle function, defined by* (2.1) *and satisfying the restriction on the parametrization* (2.9)*, for a smooth geometric flow governed by* (1.3)–(1.7)*. Then, under the assumptions in Proposition 6.12, for any $\beta \in (0,1/2)$, $t_0 > 0$ and $k \in \mathbb{N}\cup\{0\}$, there exist $c_{18}, c_{19}, c_{20}, c_{21} > 0$ such that*

$$\sum_{j=1}^{3} \|\Theta^{(j)}(\cdot,t) - \Theta_0^{(j)}\|_{C_x^\beta([0,1])} \leq c_{18} t^{\frac{1}{4}-\frac{\beta}{2}} \quad \text{for } t \in [0,1], \quad \sum_{j=1}^{3} \|\Theta^{(j)}\|_{C_{x,t}^k([0,1]\times[t_0,\infty))} \leq c_{19} \tag{6.35}$$

and

$$\sum_{j=1}^{3} |L^{(j)}(t) - L^{(j)}(0)| + |\alpha^{(j)}(t) - \alpha_0^{(j)}| \leq c_{20} t^{\frac{3}{4}} \quad \text{for } t \in [0,1], \quad \sum_{j=1}^{3} \|L^{(j)}\|_{C_t^k([t_0,\infty))} + \|\alpha^{(j)}\|_{C_t^k([t_0,\infty))} \leq c_{21}. \tag{6.36}$$

Proof. Let M and c be positive constants independent of $\Theta^{(j)}$ and will be re-chosen as necessary through this proof. Due to the restriction (2.9), we can see that

$$s = s_t^{(j)}(x) = L^{(j)}(t)x \quad \text{for } x \in [0,1], \tag{6.37}$$

where $s = s_t^{(j)}$ is the arc-length parameter of $\Gamma_t^{(j)}$. Since the right hand side of (6.26) is uniformly bounded in t and $\sigma^{(j)} \geq \sigma(0)$ holds, we have

$$\|\partial_s \kappa_t^{(j)}\|_{L^2} \leq \frac{M}{t^{1/2}}, \quad \|\partial_s^2 \kappa_t^{(j)}\|_{L^2} \leq \frac{M}{t} \quad \text{for } t > 0, \ j \in \{1,2,3\}. \tag{6.38}$$

Here, $\|\cdot\|_{L^2}$ means that the L^2-norm on $\Gamma_t^{(j)}$ with respect to the arc-length. Furthermore, by means of Proposition 5.13 and Proposition 6.10, we have

$$
\begin{aligned}
&\|\partial_s\kappa_t^{(j)}\|_{L^\infty} \le M(\|\partial_s^2\kappa_t^{(j)}\|_{L^2}^{\frac{3}{4}} + \|\kappa_t^{(j)}\|_{L^2}^{\frac{3}{4}})\|\kappa_t^{(j)}\|_{L^2}^{\frac{1}{4}} \le Mt^{-\frac{3}{4}}e^{-ct},\\
&\|\kappa_t^{(j)}\|_{L^\infty} \le M(\|\partial_s\kappa_t^{(j)}\|_{L^2}^{\frac{1}{2}} + \|\kappa_t^{(j)}\|_{L^2}^{\frac{1}{2}})\|\kappa_t^{(j)}\|_{L^2}^{\frac{1}{2}} \le Mt^{-\frac{1}{4}}e^{-ct}
\end{aligned} \tag{6.39}
$$

for any $t > 0$ and $j \in \{1,2,3\}$.

First we prove

$$
\sum_{j=1}^{3}\|\Theta^{(j)}(\cdot,t) - \Theta_0^{(j)}\|_{L^\infty} \le M(\min\{t,1\})^{\frac{1}{4}} \quad \text{for } \ t \ge 0. \tag{6.40}
$$

Re-write the first identity in (2.4) to obtain

$$
\begin{aligned}
\partial_t\Theta^{(j)}(x,t) &= \sigma^{(j)}\partial_s\kappa_t^{(j)}(s_t^{(j)}(x),t) + x\kappa_t^{(j)}(s_t^{(j)}(x),t)\lambda_t^{(j)}\lfloor_{\text{at }\vec{a}}\\
&\quad + \sigma^{(j)}\kappa_t^{(j)}(s_t^{(j)}(x),t)\left\{\int_0^{s_t^{(j)}(x)}(\kappa_t^{(j)}(s,t))^2\,ds - x\int_{\Gamma_t^{(j)}}(\kappa_t^{(j)}(s,t))^2\,ds\right\},
\end{aligned} \tag{6.41}
$$

which implies, due to (5.9), (6.39) and the boundedness of $\|\kappa_t^{(j)}\|_{L^2}$,

$$
\begin{aligned}
\|\Theta^{(j)}(\cdot,t) - \Theta_0^{(j)}\|_{L^\infty} &\le \int_0^t \|\partial_t\Theta^{(j)}(\cdot,\tilde{t})\|_{L^\infty}\,d\tilde{t}\\
&\le M\int_0^t \|\partial_s\kappa_{\tilde{t}}^{(j)}\|_{L^\infty} + \|\kappa_{\tilde{t}}^{(j)}\|_{L^\infty}\cdot\sum_{k=1}^{3}\|\kappa_{\tilde{t}}^{(k)}\|_{L^\infty} + \|\kappa_{\tilde{t}}^{(j)}\|_{L^\infty}\|\kappa_{\tilde{t}}^{(j)}\|_{L^2}^2\,d\tilde{t}\\
&\le M\int_0^t (\tilde{t}^{-\frac{3}{4}} + \tilde{t}^{-\frac{1}{2}} + \tilde{t}^{-\frac{1}{4}})e^{-c\tilde{t}}\,d\tilde{t} \le M(\min\{t,1\})^{\frac{1}{4}}.
\end{aligned}
$$

By means of (6.41) and $\partial_x\Theta^{(j)} = \kappa^{(j)}/L^{(j)}$, we can see that $\partial_t^n\partial_x^m\Theta^{(j)}$ can be represented by a polynomial in the derivatives of the curvatures with respect to the arc-length and the time derivatives of the misorientations/lengths for any $n,m \in \mathbb{N}\cup\{0\}$ except $n = m = 0$ since we already derived the formulation of $\lambda_t^{(j)}\lfloor_{\text{at }\vec{a}}$ in (2.7). Notice that the second estimate in (6.36) follows from (1.4), (6.10) and (6.17) since the derivatives of the curvatures are bounded away from $t = 0$ as in Proposition 6.12. Therefore, the second estimate in (6.35) also can be obtained easily.

We thus next prove the first estimate in (6.35). If $t \le |x-y|^2$ and $t \le 1$, we have by (6.40)

$$
|(\Theta^{(j)}(x,t) - \Theta_0^{(j)}(x)) - (\Theta^{(j)}(y,t) - \Theta_0^{(j)}(y))| \le Mt^{\frac{1}{4}} \le Mt^{\frac{1}{4}-\frac{\beta}{2}}|x-y|^{\beta}.
$$

It thus sufficient to consider the case $|x-y|^2 \le t$. Since the length $L^{(j)}$ is uniformly positive and bounded, by means of the Sobolev inequality and (6.37), we have

$$
\begin{aligned}
|\partial_s^n\kappa_t^{(j)}(s(x),t) - \partial_s^n\kappa_t^{(j)}(s(y),t)| &\le M(\|\partial_s^n\kappa_t^{(j)}\|_{L^2} + \|\partial_s^{n+1}\kappa_t^{(j)}\|_{L^2})|s(x)-s(y)|^{\frac{1}{2}}\\
&\le M(\|\partial_s^n\kappa_t^{(j)}\|_{L^2} + \|\partial_s^{n+1}\kappa_t^{(j)}\|_{L^2})|x-y|^{\frac{1}{2}}
\end{aligned}
$$

for any $n \in \mathbb{N}\cup\{0\}$. It can be applied to obtain, due to (6.41),

$$
\begin{aligned}
&|\partial_t\Theta^{(j)}(x,t) - \partial_t\Theta^{(j)}(y,t)|\\
&\le \sigma^{(j)}|\partial_s\kappa_t^{(j)}(s(x),t) - \partial_s\kappa_t^{(j)}(s(y),t)|
\end{aligned}
$$

$$
\begin{aligned}
&+|x-y|\cdot|\kappa_t^{(j)}(s(x),t)\lambda_t^{(j)}\lfloor_{\mathrm{at}\ \vec{a}}|+|y\lambda_t^{(j)}\lfloor_{\mathrm{at}\ \vec{a}}|\cdot|\kappa_t^{(j)}(s(x),t)-\kappa_t^{(j)}(s(y),t)|\\
&+\sigma^{(j)}|\kappa_t^{(j)}(s(x),t)-\kappa_t^{(j)}(s(y),t)|\cdot\left|\int_0^{s_t^{(j)}(x)}(\kappa_t^{(j)}(s,t))^2\,ds-x\int_{\Gamma_t^{(j)}}(\kappa_t^{(j)}(s,t))^2\,ds\right|\\
&+\sigma^{(j)}|\kappa_t^{(j)}(s(y),t)|\cdot\left|\int_{s_t^{(j)}(y)}^{s_t^{(j)}(x)}(\kappa_t^{(j)}(s,t))^2\,ds-(x-y)\int_{\Gamma_t^{(j)}}(\kappa_t^{(j)}(s,t))^2\,ds\right|\\
\le{}& M\Big\{(\|\partial_s\kappa_t^{(j)}\|_{L^2}+\|\partial_s^2\kappa_t^{(j)}\|_{L^2})|x-y|^{\frac12}\\
&+\left(\sum_{k=1}^{3}\|\kappa_t^{(k)}\|_{L^\infty}\right)\{\|\kappa_t^{(j)}\|_{L^\infty}|x-y|+(\|\kappa_t^{(j)}\|_{L^2}+\|\partial_s\kappa_t^{(j)}\|_{L^2})|x-y|^{\frac12}\}\\
&+(\|\kappa_t^{(j)}\|_{L^2}+\|\partial_s\kappa_t^{(j)}\|_{L^2})\|\kappa_t^{(j)}\|_{L^2}^2|x-y|^{\frac12}+\|\kappa_t^{(j)}\|_{L^\infty}(\|\kappa_t^{(j)}\|_{L^\infty}^2+\|\kappa_t^{(j)}\|_{L^2}^2)|x-y|,
\end{aligned}
$$

which implies, applying (6.38), (6.39) and the boundedness of $\|\kappa_t^{(j)}\|_{L^2}$,

$$
|\partial_t\Theta^{(j)}(x,t)-\partial_t\Theta^{(j)}(y,t)|\le M\{(1+t^{-\frac12}+t^{-1})|x-y|^{\frac12}+(t^{-\frac14}+t^{-\frac12}+t^{-\frac34})|x-y|\}\le Mt^{-\frac{3+2\beta}{4}}|x-y|^{\beta}.
$$

Here $t\le 1$ and $t\ge|x-y|^2$ have been used. Since $\frac{3+2\beta}{4}<1$, we thus obtain

$$
|(\Theta^{(j)}(x,t)-\Theta_0^{(j)}(x))-(\Theta^{(j)}(y,t)-\Theta_0^{(j)}(y))|\le\int_0^t|\partial_t\Theta^{(j)}(x,\tilde t)-\partial_t\Theta^{(j)}(y,\tilde t)|\,d\tilde t\le Mt^{\frac14-\frac{\beta}{2}}|x-y|^{\beta}.
$$

Combining (6.40), we have the first estimate in (6.35).

For the continuity of $L^{(j)}$ at $t=0$ as in (6.36) can be obtained applying the boundedness of $\|\kappa^{(j)}\|_{L^2}$ and (6.39) to (6.11). The continuity of $\alpha^{(j)}$ at $t=0$ also follows from the exponential decay of $\partial_t\alpha^{(j)}$ as in (4.14). □

We finally proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let

$$
m:=\frac{m_3}{2},\quad \varepsilon:=\frac{\min\{\varepsilon_4,\varepsilon_5\}}{2},
$$

where m_3,ε_4 and ε_5 are constants obtained in Lemma 4.2, Proposition 5.13 and Proposition 6.12, respectively. We first define the angle function $\Theta_0^{(j)}$ satisfying the restriction on the parametrization (2.9) by (2.1). Since we assume that $\Gamma_0^{(j)}$ is of class H^2 and satisfies (1.8), by means of the Sobolev inequality and $\partial_x\Theta_0^{(j)}=L^{(j)}(0)\kappa_t^{(j)}$, we then see that $\Theta_0^{(j)}\in C^{\frac12}([0,1])\cap H^1(0,1)$ and the pair of $\vec{\Theta}_0$ and $\vec{\alpha}_0$ satisfy the compatibility condition of order 0 for (2.12), namely,

$$
\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)\cos\Theta_0^{(j)}(1)=\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)\sin\Theta_0^{(j)}(1)=0.
$$

Notice also that, since (1.7) is satisfied at $t=0$, the parametrization $\xi_0^{(j)}$ of $\Gamma_0^{(j)}$ is of form

$$
\xi_0^{(j)}(x)=\left(\int_0^x L_0^{(j)}\cos\Theta_0^{(j)}(\tilde x)\,d\tilde x,\int_0^x L_0^{(j)}\sin\Theta_0^{(j)}(\tilde x)\,d\tilde x\right)+P^{(j)},\tag{6.42}
$$

where $L_0^{(j)}$ is the length of $\Gamma_0^{(j)}$, and we also have

$$
\left(\int_0^1 L_0^{(j)}\cos\Theta_0^{(j)}(x)\,dx,\int_0^1 L_0^{(j)}\sin\Theta_0^{(j)}(x)\,dx\right)+P^{(j)}
$$

$$= \left(\int_0^1 L_0^{(k)} \cos \Theta_0^{(k)}(x)\, dx, \int_0^1 L_0^{(k)} \sin \Theta_0^{(k)}(x)\, dx \right) + P^{(k)}$$

for any $j, k \in \{1,2,3\}$ since $\cup_{j=1}^3 \Gamma_0^{(j)}$ is a triod. Therefore, we can approximate $\vec{\Theta}_0$ and $\vec{L}_0$ by $\vec{\Theta}_{0,\varepsilon'} \in (C^\infty([0,1]))^3$ and $\vec{L}_{0,\varepsilon'}$, respectively, so that

(i) $\Theta_{0,\varepsilon'}^{(j)} \to \Theta_0^{(j)}$ in $C^\beta([0,1])$ and $H^1(0,1)$, and $L_{0,\varepsilon'}^{(j)} \to L_0^{(j)}$ as $\varepsilon' \to 0$ for any $j \in \{1,2,3\}$.

(ii) The pair of $\vec{\Theta}_{0,\varepsilon'}$ and $\vec{\alpha}_0$ satisfies the compatibility condition of any order $k \in \mathbb{N}$ for (2.12).

(iii) $\vec{\Theta}_{0,\varepsilon'}$ and $\vec{L}_{0,\varepsilon'}$ satisfy, for any $j, k \in \{1,2,3\}$,

$$\left(\int_0^1 L_{0,\varepsilon'}^{(j)} \cos \Theta_{0,\varepsilon'}^{(j)}(x)\, dx, \int_0^1 L_{0,\varepsilon'}^{(j)} \sin \Theta_{0,\varepsilon'}^{(j)}(x)\, dx \right) + P^{(j)}$$
$$= \left(\int_0^1 L_{0,\varepsilon'}^{(k)} \cos \Theta_{0,\varepsilon'}^{(k)}(x)\, dx, \int_0^1 L_{0,\varepsilon'}^{(k)} \sin \Theta_{0,\varepsilon'}^{(k)}(x)\, dx \right) + P^{(k)}.$$

Note that, since we only need to adjust the approximation near the boundary to satisfy (ii), the conditions (i), (ii) and (iii) are compatible even if the pair of $\vec{\Theta}_0$ and $\vec{\alpha}_0$ satisfies the compatibility condition of order 0 only. Constructing an approximated initial curve $\Gamma_{0,\varepsilon'}^{(j)}$ by the formula as in (6.42) from $\Theta_{0,\varepsilon'}^{(j)}$ and $L_{0,\varepsilon'}$, we then see that the pair of $\{\Gamma_{0,\varepsilon'}^{(j)}\}_{j\in\{1,2,3\}}$ and $\vec{\alpha}_0$ satisfies the assumptions in Theorem 1.1 with any $k \geq 3$ and hence we obtain a smooth geometric flow governed by (1.3)–(1.7) starting from the approximated initial datum. Let $\{\Gamma_{t,\varepsilon'}^{(j)}\}_{j\in\{1,2,3\}}$ and $\{\alpha_{\varepsilon'}^{(j)}(t)\}_{j\in\{1,2,3\}}$ be the moving triod and the set of misorientations of the smooth geometric flow, respectively. Since the length of $\Gamma_{0,\varepsilon'}^{(j)}$ is $L_{0,\varepsilon'}^{(j)}$ and the curvature $\kappa_{0,\varepsilon'}^{(j)}$ of $\Gamma_{0,\varepsilon'}^{(j)}$ is $\partial_s \Theta_{0,\varepsilon'}^{(j)} = \partial_x \Theta_{0,\varepsilon'}^{(j)} / L_{0,\varepsilon'}$, we may see that

$$\sum_{j=1}^3 \sigma(\Delta^{(j)}\alpha_0) L_{0,\varepsilon'}^{(j)} \leq \sigma(0) m_3, \quad \sum_{j=1}^3 \left\{ \left(\Delta^{(j)}\alpha_0\right)^2 + \int_{\Gamma_{0,\varepsilon'}^{(j)}} \left(\sigma(\Delta^{(j)}\alpha_0)\right)^2 (\kappa_{0,\varepsilon'}^{(j)})^2\, ds \right\} \leq \min\{\varepsilon_4, \varepsilon_5\}$$

is satisfied for small $\varepsilon' > 0$ by means of the convergence as in (i). Therefore, all estimates in Section 4–Section 6 can be applied to the smooth geometric flow starting from the approximated initial datum to extend the maximum existence time. We summarize the estimates as follows.

(a) By means of Lemma 4.1 and Lemma 4.2, the length $L_{\varepsilon'}^{(j)}$ of $\Gamma_{\varepsilon'}^{(j)}$ is uniformly positive and bounded from above.

(b) By means of Lemma 4.9, the angle function $\Theta_{\varepsilon'}^{(j)}$ of $\Gamma_{t,\varepsilon'}^{(j)}$ satisfy $\Theta_{\varepsilon'}^{(j+1)} - \Theta_{\varepsilon'}^{(j)} \in (0,\pi)$ at the junction point $\vec{a}_{\varepsilon'}(t)$ of the moving triod $\{\Gamma_{t,\varepsilon'}^{(j)}\}_{j\in\{1,2,3\}}$ for any $j \in \{1,2,3\}$ and $t \geq 0$.

(c) The a priori estimates as in Corollary 6.14 holds.

Therefore, we can apply Proposition 2.14 to conclude that the flow exists globally in time.

Furthermore, applying the Arzelá-Ascoli theorem, there exists a solution $\vec{\Theta} \in (C^\infty([0,1]\times(0,\infty)))^3$, $\vec{L} \in (C^\infty((0,\infty)))^3$, $\vec{\alpha} \in (C^\infty((0,\infty)))^3$ to (2.12) such that $\vec{\Theta}_{\varepsilon'}, \vec{L}_{\varepsilon'}, \vec{\alpha}_{\varepsilon'}$ sub-converges to the solution in $(C^\infty_{\mathrm{loc}}([0,1]\times(0,\infty)))^3 \times (C^\infty_{\mathrm{loc}}((0,\infty)))^3 \times (C^\infty_{\mathrm{loc}}((0,\infty)))^3$. Then, we can construct a smooth geometric flow governed by (1.3)–(1.7) from the solution to (2.12) by the formulation as (6.42), and we can also see that each curve $\Gamma_t^{(j)}$ of the smooth geometric flow converges to the original initial curve $\Gamma_0^{(j)}$ as $t \to 0$ in the $C^{1+\beta}$ topology, for any $\beta \in (0,1/2)$, due to the convergence property (i) and the continuities as in (6.35) and (6.36). On the other hand, since the exponential L^2-decay of the derivatives of the curvatures can be obtained by a similar estimate for (6.39) away from $t = 0$, by means of the uniqueness of the stationary solution as in Proposition 3.1 and the exponential decay of $\Delta^{(j)}\alpha$, as $t \to \infty$, each curve $\Gamma_t^{(j)}$ converges to a line segment of the unique Steiner triod in C^∞ topology and $\alpha^{(j)}(t)$ converges to $(\alpha_0^{(1)} + \alpha_0^{(2)} + \alpha_0^{(3)})/3$ due to the preservation of $\sum_{j=1}^3 \alpha^{(j)}(t)$. □

Remark 6.15. (i) According to Remark 6.13, the boundedness of

$$\|\Theta^{(j)}\|_{C^k_{x,t}([0,1]\times[t_0,T])} + \|L^{(j)}\|_{C^k_t([t_0,T])} + \|\alpha^{(j)}\|_{C^k_t([t_0,T])},$$

for each $k \in \mathbb{N}, 0 < t_0 < T$, can be obtained if the properties (a)–(b) in Remark 6.13 hold for $t \leq T$ and $j \in \{1,2,3\}$. Therefore, a smooth geometric flow can be constructed if the initial triod is of class H^2. Furthermore, if its maximum existence time T' is finite, then either (a), (b) or (c) is lost at the time T'.

(ii) In the study [34] for the classical curvature flow, the regularity estimates of the parametrization $\xi^{(j)}$ are derived from the energy type estimates for not only the curvature but also a restricted tangent velocity $\lambda_t^{(j)} = \langle \frac{\partial_x^2 \xi^{(j)}}{|\partial_x \xi^{(j)}|^2}, \tau_t^{(j)} \rangle$. In our method, we does not require the energy type estimate of the tangent velocity and, additionally, some discussion to conclude that $|\partial_x \xi^{(j)}| \neq 0$ if $L^{(j)} > 0$, which is not obvious for the re-parametrization in [34], is not necessary in our method.

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