

ON QUOTIENTS OF BOUNDED HOMOGENEOUS DOMAINS BY UNIPOTENT DISCRETE GROUPS

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ABSTRACT. We show that the quotient of any bounded homogeneous domain by a unipotent discrete group of automorphisms is holomorphically separable. Then we give a necessary condition for such a quotient to be Stein and prove that in some cases this condition is also sufficient.

1. INTRODUCTION

Given a Lie group of holomorphic transformations of a Stein space X , one would like to have a complex quotient space whose holomorphic functions are the invariant holomorphic functions on X and which is again Stein. In the case that the Lie group is *compact*, it is possible to average holomorphic functions over this group and to construct in this way an invariant-theoretic Stein quotient space, see [15]. On the other hand, when an infinite discrete group acts properly by holomorphic transformations on a Stein space, then the orbit space is again a complex space. However, in general there is no averaging method in order to construct holomorphic functions on this quotient space. In fact, it is not hard to find examples where the quotient space is compact, hence where all invariant holomorphic functions are constant.

There are numerous results in the literature where certain quotients of Stein manifolds by proper actions of infinite discrete groups are shown to be holomorphically separable or Stein. In [9] the authors showed that quotients of the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ by proper \mathbb{Z} -actions are Stein, which was then generalized to simply-connected bounded domains of holomorphy in \mathbb{C}^2 ([26]), to arbitrary bounded homogeneous domains in \mathbb{C}^n ([24]), and to Akhiezer-Gindikin domains ([35]). Quotients of the unit ball and Akhiezer-Gindikin domains by discrete groups that act cocompactly on a real form of these domains are shown to be Stein in [4, Proposition 6.4] and [3, Corollary 7], respectively. An analog result is shown in [23] for quotients of complex solvable Lie groups by discrete subgroups that act cocompactly on a real form having purely imaginary spectrum. Quotients of complex Olshanski semigroups by certain discrete groups were studied in [1] and [25]. In [6] actions of discrete groups on Kähler-Hadamard manifolds and their quotients are investigated. Recently, quotients of the unit ball by certain convex-cocompact discrete groups were studied from the viewpoint of complex-hyperbolic geometry in [7]. In [27] Schottky group actions on the unit ball having Stein quotients are constructed. Most of these results strongly rely on Lie theory.

In this paper, we are concerned with the action of a *unipotent* discrete group Γ of holomorphic automorphisms of a bounded homogeneous domain. In order to state the main results, let $D \subset \mathbb{C}^n$ be a bounded domain and recall that its automorphism group $\text{Aut}(D)$ is a real Lie group that acts properly on D , see [5]. Let G be the connected component of $\text{Aut}(D)$ that contains the identity. The domain D is *homogeneous* if $\text{Aut}(D)$ and hence G act transitively on it. Let us fix a base point $p_0 \in D$. Its isotropy group $K := G_{p_0}$ is a maximal compact subgroup of G and there exists a decomposition $G = KR$ where R is a simply-connected

split solvable Lie group, see [20] and [32]. Consequently, R is isomorphic to a semi-direct product $A \ltimes N$ where N is the nilradical of R and $A \cong (\mathbb{R}^{>0})^r$.

Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain and let $G = KAN$ be the decomposition of $G = \text{Aut}^0(D)$ introduced above. A discrete subgroup of G will be called *unipotent* if it is conjugate to a subgroup of N . The research presented in this paper was motivated by the following result.

Theorem 1.1. *Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain and let Γ be a unipotent discrete group of automorphisms of D . Then the complex manifold D/Γ is holomorphically separable.*

It is therefore natural to ask under which additional conditions on Γ the quotient manifold D/Γ is Stein. Suppose from now on that Γ is a discrete subgroup of N . It is well known that the simply-connected nilpotent group N admits a unique structure as a real-algebraic group such that its Zariski closed subgroups are precisely its connected Lie subgroups, see [33, Chapter 2.4.2]. Hence, consider the Zariski closure N_Γ of Γ in N . Then N_Γ is a simply-connected nilpotent Lie group such that N_Γ/Γ is compact.

We have the following necessary condition for D/Γ to be Stein. Note that to the best of my knowledge it is not known whether this condition is also sufficient.

Proposition 1.2. *Let $\Gamma \subset N$ be a discrete subgroup and consider its Zariski closure N_Γ . If D/Γ is Stein, then all N_Γ -orbits in D are totally real.*

Since $R \cong A \ltimes N$ acts simply transitively on D , we can identify its Lie algebra $\mathfrak{r} = \mathfrak{a} \oplus \mathfrak{n}$ with $T_{p_0}D = \mathbb{C}^n$ and thus obtain a complex structure $j \in \text{End}(\mathfrak{r})$. Let N_Γ be the Zariski closure of a discrete subgroup $\Gamma \subset N$ and let \mathfrak{n}_Γ be its Lie algebra. The orbit $N_\Gamma \cdot p_0$ is totally real if and only if $\mathfrak{n}_\Gamma \cap j(\mathfrak{n}_\Gamma) = \{0\}$, in which case we call \mathfrak{n}_Γ a *totally real subalgebra* of \mathfrak{n} . If \mathfrak{n}_Γ is totally real and if the real dimension of \mathfrak{n}_Γ coincides with the complex dimension of D , then we say that \mathfrak{n}_Γ is a *maximal totally real subalgebra* of \mathfrak{n} . We have the following sufficient criterion for D/Γ to be Stein.

Proposition 1.3. *Let $\Gamma \subset N$ be a discrete subgroup and consider its Zariski closure N_Γ . If \mathfrak{n}_Γ is contained in a maximal totally real subalgebra of \mathfrak{n} , then D/Γ is Stein.*

The methods and results described above allow us to prove the main result of this paper, which in particular answers the question raised in [6, Remark 7.6] in the negative.

Theorem 1.4. *Let D be the unit ball or the Lie ball and let Γ be a unipotent discrete group of automorphisms of D having Zariski closure $N_\Gamma \subset N$. Then D/Γ is Stein if and only if \mathfrak{n}_Γ is totally real.*

The proof of Theorem 1.4 relies in part on the fact that for the unit ball and the Lie ball the necessary condition given in Proposition 1.2 is indeed sufficient as well.

Let us outline the structure of this paper. In Section 2 we review some parts of the structure theory of bounded homogeneous domains and prove Theorem 1.1 as well as Propositions 1.2 and 1.3. Sections 3 and 4 contain the proves of Theorem 1.4 for the unit ball and the Lie ball, respectively. In the last section 5 we present an example that shows that Theorem 1.4 does *not* hold true for arbitrary bounded homogeneous domains.

Acknowledgments. I would like to thank Karl Oeljeklaus for helpful discussions. Financial support by the ANR-project QuaSiDy (ANR-21-CE40-0016-01) is gratefully acknowledged.

2. RESULTS FOR ARBITRARY BOUNDED HOMOGENEOUS DOMAINS

We begin by reviewing the structure of the automorphism group of a bounded homogeneous domain, its unbounded realization as a Siegel domain, and its associated normal

j -algebra. As main references we refer the reader to [20] and [30]. We then show that the quotient of a bounded homogeneous domain by a unipotent discrete group Γ of automorphisms is holomorphically separable, see Theorem 1.1, and prove Propositions 1.2 and 1.3.

2.1. Bounded homogeneous domains and normal j -algebras. Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain with base point $p_0 \in D$ and consider the decomposition $G = KR$ where $G = \text{Aut}^0(D)$ and $K = G_{p_0}$. The elements of the Lie algebra \mathfrak{g} will be viewed as complete holomorphic vector fields on D . Since the group R is split solvable, we have $R \cong A \ltimes N$ where N is the nilradical of R . Moreover, the adjoint representation of the Abelian group A on \mathfrak{n} is diagonalizable over \mathbb{R} , compare [20, Proposition 2.8].

It is well known that every bounded homogeneous domain $D \subset \mathbb{C}^n$ is biholomorphic to a Siegel domain of the second kind $\widehat{D} \subset \mathbb{C}^n$, see [34].

Example. The unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is biholomorphically equivalent to

$$\widehat{\mathbb{B}}_n = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}; \text{Im}(z) - \|w\|^2 > 0\},$$

the Lie ball $\mathbb{L}_n \subset \mathbb{C}^n$ is biholomorphic to

$$\widehat{\mathbb{L}}_n = \{z \in \mathbb{C}^n; \text{Im}(z_n)^2 - (\text{Im}(z_1)^2 + \cdots + \text{Im}(z_{n-1})^2) > 0, \text{Im}(z_n) > 0\},$$

and the Siegel disk $\mathbb{S}_n \subset \text{Sym}(n, \mathbb{C})$ may be realized as Siegel's upper half-plane

$$\widehat{\mathbb{S}}_n = \{Z \in \text{Sym}(n, \mathbb{C}); \text{Im}(Z) \text{ is positive definite}\}.$$

In Helgason's notation, see [16, Table V, Chapter X], the unit ball, the Lie ball and the Siegel disk correspond to the Hermitian symmetric spaces of types $AIII(p = 1, q = n)$, $BDI(p = 2, q = n)$ and CI , respectively.

The split-solvable group $R \cong A \ltimes N$ acts by affine transformations on \widehat{D} . In particular, this R -action extends to the whole of \mathbb{C}^n . It follows from the explicit realization of its affine automorphism group, see e.g. [20, Chapters 2 and 3], that N and hence $N^{\mathbb{C}}$ act *algebraically* by affine transformations on \mathbb{C}^n and that every transformation in $N^{\mathbb{C}}$ has Jacobi determinant equal to 1.

Since R acts simply transitively on D , we may identify its Lie algebra $\mathfrak{r} = \mathfrak{a} \oplus \mathfrak{n}$ with $T_{p_0}D = \mathbb{C}^n$ and thus obtain an integrable complex structure $j \in \text{End}(\mathfrak{r})$.

In order to show that (\mathfrak{r}, j) is a normal j -algebra we need to prove the existence of a linear form $\lambda \in \mathfrak{r}^*$ such that for all $x, y \in \mathfrak{r}$ we have

$$\lambda[j(x), j(y)] = \lambda[x, y] \quad \text{and} \quad \lambda[j(x), x] > 0 \text{ if } x \neq 0,$$

see part (III) of the definition in [30, p. 51].

In [20] and [30] existence of this linear form $\lambda \in \mathfrak{r}^*$ is deduced from [22]. Let us explain here how λ can be constructed from the Bergman metric of D via a moment map. (Recall that the Bergman metric of a bounded domain $D \subset \mathbb{C}^n$ is an $\text{Aut}(D)$ -invariant Kähler metric on D . For its definition we refer the reader to [21, Chapter 4.10].) It is shown in [18] that there exists an R -equivariant holomorphic embedding of D into Siegel's upper half-plane $\widehat{\mathbb{S}}_n$. Since the domain \mathbb{S}_n is symmetric, the group $\text{Aut}^0(\mathbb{S}_n)$ is semisimple and thus there exists an equivariant moment map for the action of $\text{Aut}^0(\mathbb{S}_n)$ on \mathbb{S}_n . Pulling back this moment map to D we obtain an R -equivariant moment map $\mu: D \rightarrow \mathfrak{r}^*$ with respect to the Bergman metric ω of D . Note that μ is a diffeomorphism onto its image, an open coadjoint orbit. Moreover, μ is a symplectomorphism with respect to the Kostant-Kirillov form on this coadjoint orbit, i.e., we have

$$(2.1) \quad \omega_{p_0}(x(p_0), y(p_0)) = \mu(p_0)[x, y]$$

for all $x, y \in \mathfrak{r}$, see [13, Equations (26.2) and (26.8)]. As a consequence, we can take $\lambda := \mu(p_0) \in \mathfrak{r}^*$ in the definition of the normal j -algebra associated with D . For the structure theory of normal j -algebras we refer the reader to [30, Chapter 2].

Remark. For a connected closed subgroup R' of R the orbit $R' \cdot p_0$ is totally real if and only if $\mathfrak{r}' \cap j(\mathfrak{r}') = \{0\}$. In this case, we refer to \mathfrak{r}' as a totally real subalgebra of \mathfrak{r} . Note that every isotropic orbit (with respect to the Bergman metric) is totally real. Moreover, every orbit of an Abelian subgroup of R is totally real since $\mu(p_0)[j(x), x] > 0$ holds for $x \neq 0$. This last statement is also a consequence of the fact that D is Kobayashi-hyperbolic, since an Abelian group of automorphisms that is not totally real would yield a non-constant holomorphic map from \mathbb{C} to D .

Conversely, every abstract normal j -algebra is the normal j -algebra associated with some bounded homogeneous domain, see [30, Appendix].

2.2. Quotients by unipotent discrete groups are holomorphically separable. Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain and let Γ be a unipotent subgroup of $G = \text{Aut}^0(D)$.

Proof of Theorem 1.1. Any unipotent discrete subgroup $\Gamma \subset G$ acts freely and properly on $D \times \mathbb{C}$ by

$$\gamma \cdot (z, t) := (\gamma(z), t \det d\gamma(z)^{-1}),$$

so that we have the quotient manifold $L := D \times_{\Gamma} \mathbb{C} := (D \times \mathbb{C})/\Gamma$. Moreover, the projection onto the first factor $D \times \mathbb{C} \rightarrow D$ is Γ -equivariant and the induced map $L = D \times_{\Gamma} \mathbb{C} \rightarrow D/\Gamma$ defines a holomorphic line bundle on D/Γ . A holomorphic section in L corresponds to a Γ -equivariant holomorphic function from D to $D \times \mathbb{C}$. Holomorphic sections in the k -fold tensor product $L^{\otimes k}$ can be constructed via Poincaré series for $k \geq 2$ and separate the points of D/Γ in the following sense. For every pair of elements $p, q \in D/\Gamma$ with $p \neq q$ there exist $k \geq 2$ (depending on p and q) and a holomorphic section s in $L^{\otimes k}$ such that $s(p) = 0$ and $s(q) \neq 0$, see [30, Lemma 3.4.1]. We will finish the proof by showing that for a unipotent discrete subgroup Γ the line bundle L admits a non-vanishing holomorphic section. It then follows that L , and thus all of its powers, are holomorphically trivial.

In order to do so, let $\varphi: D \rightarrow \widehat{D}$ be a Γ -equivariant biholomorphic map to a Siegel domain of the second kind on which Γ acts by affine transformations having Jacobi determinant 1. In other words, we have $\varphi \circ \gamma = \widehat{\gamma} \circ \varphi$ where $\det d\widehat{\gamma}(z) = 1$ for all $z \in \widehat{D}$. Then, the chain rule implies

$$\begin{aligned} \det d\varphi(\gamma(z)) &= \det d\gamma(z)^{-1} \cdot \det d(\varphi \circ \gamma)(z) \\ &= \det d\gamma(z)^{-1} \cdot \det d(\widehat{\gamma} \circ \varphi)(z) = \det d\gamma(z)^{-1} \cdot \det d\varphi(z). \end{aligned}$$

Consequently, the holomorphic map $s: D \rightarrow D \times \mathbb{C}^*$ given by $s(z, t) = (z, \det d\varphi(z))$ is Γ -equivariant and thus defines a non-vanishing holomorphic section in L , as desired. \square

2.3. A necessary condition for Steinness. In this subsection we prove Proposition 1.2.

An important ingredient for the proof is the following result of Loeb. Let $\Gamma \subset G$ be a discrete group of unipotent automorphisms of D . After conjugation we may suppose that Γ is contained in N . Let N_{Γ} be the real Zariski closure of Γ in N and let $N_{\Gamma}^{\mathbb{C}}$ be its universal complexification. Then the complex homogeneous space $N_{\Gamma}^{\mathbb{C}}/\Gamma$ is Stein, see [11] and [23, Théorème 1].

Recall that the complexification $N^{\mathbb{C}}$ is a unipotent complex algebraic group and its action on the Siegel domain \widehat{D} extends to an algebraic action on \mathbb{C}^n . In particular, the orbits of any algebraic subgroup of $N^{\mathbb{C}}$ are closed in \mathbb{C}^n , see [2, Proposition 4.10].

Lemma 2.1. *The orbit $N_\Gamma \cdot z$ is totally real in \widehat{D} if and only if the isotropy group $(N_\Gamma^{\mathbb{C}})_z$ is trivial.*

Proof. For this, let $x + iy \in \mathfrak{n}_\Gamma \oplus i\mathfrak{n}_\Gamma = \mathfrak{n}_\Gamma^{\mathbb{C}}$ be a holomorphic vector field on \widehat{D} and consider $x(z) + j_z y(z) \in T_z \widehat{D}$ where j_z is the complex structure of $T_z \widehat{D}$, for some $z \in \widehat{D}$. Suppose that $N_\Gamma \cdot z$ is totally real. Then $x(z) + j_z y(z) = 0$ implies $x(z) = y(z) = 0$, and since N_Γ acts freely on \widehat{D} , we obtain $x = y = 0$. Consequently, the isotropy $(N_\Gamma^{\mathbb{C}})_z$ is discrete. Since $N_\Gamma^{\mathbb{C}}$ acts algebraically and has no finite subgroups, we conclude that $(N_\Gamma^{\mathbb{C}})_z$ is trivial.

Conversely, if $N_\Gamma \cdot z$ is not totally real for some $z \in \widehat{D}$, then there are $x, y \in \mathfrak{n}_\Gamma$ such that $x(z) = j_z y(z)$. Thus the vector field $x - iy \in \mathfrak{n}_\Gamma^{\mathbb{C}}$ vanishes at z , i.e., $N_\Gamma^{\mathbb{C}}$ does not act freely. \square

We are now in position to prove the Proposition 1.2.

Proof of Proposition 1.2. Suppose that D/Γ is a Stein manifold. As a first step we are going to show that N_Γ has at least one totally real orbit in D .

Since D/Γ is Stein, the domain D admits a Γ -invariant smooth strictly plurisubharmonic function ρ that is exhaustive modulo Γ . We may assume that 0 is a global minimum of ρ . Since N_Γ/Γ is compact, we can suppose without loss of generality that the function ρ is invariant under N_Γ and is an exhaustion modulo N_Γ , see [23, Lemme 2.1]. Let $N_\Gamma \cdot z$ be an orbit lying in the minimal set of ρ . Then $N_\Gamma \cdot z$ is totally real due to [14].

Finally, due to Lemma 2.1, it is enough to show that $N_\Gamma^{\mathbb{C}}$ acts freely on $N_\Gamma^{\mathbb{C}} \cdot \widehat{D} \subset \mathbb{C}^n$. In order to do this, let $z \in \widehat{D}$ and consider the algebraic subgroup $H := (N_\Gamma^{\mathbb{C}})_z$ of $N_\Gamma^{\mathbb{C}}$. Since $N_\Gamma^{\mathbb{C}} \cdot z$ is closed in \mathbb{C}^n , the intersection $\Omega := \widehat{D} \cap (N_\Gamma^{\mathbb{C}} \cdot z)$ is Stein and N_Γ -equivariantly biholomorphic to an N_Γ -invariant Stein open neighborhood Ω of $N_\Gamma \cdot eH \cong N_\Gamma$ in $N_\Gamma^{\mathbb{C}}/H \cong \mathbb{C}^k$. As D/Γ is Stein by assumption, Ω/Γ is also Stein. By applying [14] and [23] as above, one sees that $N_\Gamma \cdot eH$ is totally real in $N_\Gamma^{\mathbb{C}}/H$. Then, the analog argument as in the proof of Lemma 2.1 shows that H is trivial, as wished. \square

Remark. Every orbit lying in the minimal set of ρ is isotropic with respect to the Kähler form $i\partial\bar{\partial}\rho$. In general, it is not isotropic with respect to the Bergman metric of D .

2.4. A sufficient condition for D/Γ to be Stein. In this subsection we prove Proposition 1.3.

As above, let $\Gamma \subset N$ be a discrete group of unipotent automorphisms of a bounded homogeneous domain D and let N_Γ be its Zariski closure in N . Realize D as a Siegel domain of the second kind $\widehat{D} \subset \mathbb{C}^n$ such that N acts by affine transformations on \widehat{D} .

Proof of Proposition 1.3. By assumption, there exists a maximal totally real subalgebra $\widehat{\mathfrak{n}}_\Gamma \subset \mathfrak{n}$ which contains \mathfrak{n}_Γ . At the group level we thus find a connected subgroup $\widehat{N}_\Gamma \subset N$ containing N_Γ such that $\widehat{N}_\Gamma \cdot p_0$ is a maximal totally real submanifold of $N \cdot p_0$.

Consider the complexifications $N_\Gamma^{\mathbb{C}} \subset \widehat{N}_\Gamma^{\mathbb{C}} \subset N^{\mathbb{C}}$ as well as their algebraic actions by affine transformations on \mathbb{C}^n . Since $\widehat{N}_\Gamma \cdot p_0$ is maximally totally real, the orbit $\widehat{N}_\Gamma^{\mathbb{C}} \cdot p_0$ is open in \mathbb{C}^n and the $\widehat{N}_\Gamma^{\mathbb{C}}$ -isotropy at p_0 is trivial, see Lemma 2.1. Since $\widehat{N}_\Gamma^{\mathbb{C}} \cdot p_0$ is also closed in \mathbb{C}^n , we obtain $\widehat{N}_\Gamma^{\mathbb{C}} \cdot p_0 = \mathbb{C}^n$. In other words, $\widehat{N}_\Gamma^{\mathbb{C}}$ acts freely and transitively on \mathbb{C}^n .

It follows that $N_\Gamma^{\mathbb{C}}$ acts freely and properly on $\mathbb{C}^n \cong \widehat{N}_\Gamma^{\mathbb{C}}$ and that $\mathbb{C}^n/N_\Gamma^{\mathbb{C}} \cong \mathbb{C}^m$. Therefore the corresponding holomorphic principal bundle is holomorphically trivial. Together with [23, Théorème 1] this implies that $\mathbb{C}^n/\Gamma \cong (N_\Gamma^{\mathbb{C}}/\Gamma) \times \mathbb{C}^m$ is a Stein manifold. Since \widehat{D}/Γ is a locally Stein domain in \mathbb{C}^n/Γ , the quotient \widehat{D}/Γ is likewise Stein by [8]. \square

Remark. The proof of Proposition 1.3 shows that, if \mathfrak{n}_Γ is contained in a maximal totally real subalgebra of \mathfrak{n} , then $N_\Gamma^{\mathbb{C}}$ acts properly and freely on \mathbb{C}^n . Hence, all N_Γ -orbits are totally real in D . Note that this follows also from Proposition 1.2.

Remark. Under the hypotheses of Proposition 1.3, $\widehat{N}_\Gamma^{\mathbb{C}} \cong \widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{D} = \mathbb{C}^n$ is the universal globalization of the induced local $\widehat{N}_\Gamma^{\mathbb{C}}$ -action on \widehat{D} in the sense of [28], see the proof of Proposition 1.3.

3. THE CASE OF THE UNIT BALL

In this section we consider the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. Firstly, we illustrate Theorem 1.4 by two examples that can be analyzed by ad hoc methods. Then we review the structure of the normal j -algebra \mathfrak{b}_n of \mathbb{B}_n , which will be used to show that every totally real subalgebra of \mathfrak{b}_n is contained in a maximal totally real one.

3.1. Two examples. We identify the unit ball \mathbb{B}_n with its unbounded realization

$$\widehat{\mathbb{B}}_n = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}; \operatorname{Im}(z) - \|w\|^2 > 0\}.$$

For an explicit description of the vector fields belonging to its normal j -algebra \mathfrak{b}_n as well as of the corresponding one-parameter groups we refer the reader to [24, Table 1, p. 341].

First, let us present an example of a nonabelian discrete group Γ such that \mathbb{B}_3/Γ is Stein, thus answering the question raised in [6, Remark 7.6] in the negative. Note that 3 is the smallest dimension so that a similar example can be constructed.

Example. Let us consider the complete holomorphic vector fields $x_1 := 2iw_1 \frac{\partial}{\partial z} + \frac{\partial}{\partial w_1}$, $x_2 := 2(w_1 + w_2) \frac{\partial}{\partial z} + i \frac{\partial}{\partial w_1} + i \frac{\partial}{\partial w_2}$, and $x_3 := \frac{\partial}{\partial z}$ on $\widehat{\mathbb{B}}_3$. Since their only non-vanishing Lie bracket is $[x_1, x_2] = 4x_3$, they generate a three-dimensional subalgebra in the nilradical of \mathfrak{b}_3 , isomorphic to the three-dimensional Heisenberg algebra. Moreover, this algebra is defined over \mathbb{Q} and therefore the corresponding connected subgroup admits a cocompact discrete subgroup Γ , which justifies the notation $N_\Gamma = \exp(\mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \mathbb{R}x_3)$. One verifies directly that every N_Γ -orbit in $\widehat{\mathbb{B}}_3$ is maximally totally real. This implies that the universal globalization of the local $N_\Gamma^{\mathbb{C}}$ -action on $\widehat{\mathbb{B}}_3$ is isomorphic to $N_\Gamma^{\mathbb{C}}$ and the N_Γ -action on \mathbb{B}_3 corresponds to left multiplication on $N_\Gamma^{\mathbb{C}}$. Since $N_\Gamma^{\mathbb{C}}/\Gamma$ is a Stein manifold which contains $\widehat{\mathbb{B}}_3/\Gamma$ as a domain, we see that \mathbb{B}_3/Γ is Stein due to [8] while Γ is not Abelian.

Secondly, we present an explicit example of a unipotent discrete group Γ where \mathbb{B}_2/Γ is non-Stein but admits a (singular) Stein envelope.

Example. Let Γ be the discrete group consisting of the automorphisms

$$(z, w) \mapsto (z + 2(n + im)w + i(m^2 + n^2) + 2k, w + m + in) =: \varphi_{2k, m+in}(z, w),$$

where $m, n, k \in \mathbb{Z}$ and $(z, w) \in \widehat{\mathbb{B}}_2$. Although we do not need this fact, let us remark that the Lie algebra \mathfrak{n}_Γ of the Zariski closure of Γ coincides with the nilradical of \mathfrak{b}_2 .

In order to determine the quotient $\widehat{\mathbb{B}}_2/\Gamma$, we first consider the action of the normal subgroup $\Gamma_0 := \{\varphi_{2k, 0}; k \in \mathbb{Z}\} \triangleleft \Gamma$. The map $p: \widehat{\mathbb{B}}_2 \rightarrow \mathbb{C}^* \times \mathbb{C}$ given by $p(z, w) = (e^{i\pi z}, w)$ is Γ_0 -invariant and yields

$$\widehat{\mathbb{B}}_2/\Gamma_0 \cong p(\widehat{\mathbb{B}}_2) = \{(z, w) \in \mathbb{C}^* \times \mathbb{C}; |z| < e^{-\pi|w|^2}\}.$$

The induced action of $\mathbb{Z} \oplus i\mathbb{Z} \cong \Gamma/\Gamma_0$ on $p(\widehat{\mathbb{B}}_2)$ is given by

$$(m + in) \cdot (z, w) = (e^{i\pi(2(n+im)w + i(m^2+n^2))}, w + m + in).$$

It follows that the action of $\mathbb{Z} \oplus i\mathbb{Z}$ on $p(\widehat{\mathbb{B}}_2)$ extends to a proper action on the whole of \mathbb{C}^2 and that the equivariant map $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, $(z, w) \mapsto w$, induces a holomorphic line bundle $L := \mathbb{C}^2/(\mathbb{Z} \oplus i\mathbb{Z}) \rightarrow E = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. We see that $\widehat{\mathbb{B}}_2/\Gamma \cong p(\widehat{\mathbb{B}}_2)/(\mathbb{Z} \oplus i\mathbb{Z})$ embeds into L as an open neighborhood of the zero section minus this zero section. Note

that $\widehat{\mathbb{B}}_2/\Gamma$ is Kobayashi-hyperbolic since $\widehat{\mathbb{B}}_2$ is so, see [21, Theorem 3.2.8(2)]. Since $\partial\widehat{\mathbb{B}}_2$ is strictly pseudoconvex, the zero-section in L has a strictly pseudoconvex neighborhood and hence is negative in the sense of Grauert, see [12, Satz 1]. It follows that the zero section of L can be blown down to yield a Stein space Y with an isolated singularity containing $\widehat{\mathbb{B}}_2/\Gamma$ as a neighborhood of this singularity minus the singularity. Consequently, $\widehat{\mathbb{B}}_2/\Gamma$ is holomorphically separable but not Stein.

3.2. The normal j -algebra of the unit ball. The automorphism group of the unit ball \mathbb{B}_n is $G = \text{Aut}(\mathbb{B}_n) \cong \text{PSU}(n, 1)$. Let $G = KAN$ be an Iwasawa decomposition with maximal compact subgroup $K = G_{p_0} \cong \text{U}(n)$ for $p_0 = 0$. It is well known that G is of real rank 1, i.e., that $\dim A = 1$. Moreover, under the identification $T_{p_0}\mathbb{B}_n \cong \mathfrak{a} \oplus \mathfrak{n}$ we can rewrite the moment map condition (2.1) as

$$(3.1) \quad [x, y](p_0) = \omega_{p_0}(x(p_0), y(p_0))\zeta(p_0)$$

for all $x, y \in \mathfrak{n}$, compare [30, p. 52]. This means that \mathfrak{n} is the Heisenberg algebra of dimension $2n - 1$ with center $\mathbb{R}\zeta$ defined by the symplectic form ω_{p_0} induced by the Bergman metric of \mathbb{B}_n . As we shall see in the following subsection, it is this close relation between the geometry of (\mathbb{B}_n, ω) and the structure of N that enables us to prove Theorem 1.4 for the unit ball.

Recall from [24, Section 4.2] that the normal j -algebra $\mathfrak{b}_n = \mathfrak{a} \oplus \mathfrak{n}$ can be written as

$$\mathfrak{a} = \mathbb{R}\alpha \quad \text{and} \quad \mathfrak{n} = \bigoplus_{k=1}^{n-1} \mathbb{R}\xi_k \oplus \bigoplus_{k=1}^{n-1} \mathbb{R}\xi'_k \oplus \mathbb{R}\zeta,$$

where the only non-zero Lie brackets are

$$[\xi_k, \xi'_k] = \zeta, \quad [\alpha, \xi_k] = -\xi_k, \quad [\alpha, \xi'_k] = -\xi'_k, \quad \text{and} \quad [\alpha, \zeta] = -2\zeta$$

for all $1 \leq k \leq n - 1$, and where the complex structure $j: \mathfrak{b}_n \rightarrow \mathfrak{b}_n$ is given by

$$j(\zeta) = \alpha \quad \text{and} \quad j(\xi_k) = \xi'_k$$

for all $1 \leq k \leq n - 1$.

3.3. Proof of Theorem 1.4 for the unit ball. Let $\mathfrak{b}_n = \mathfrak{a} \oplus \mathfrak{n}$ be the normal j -algebra of \mathbb{B}_n . In this subsection we are going to show that every totally real subalgebra of \mathfrak{n} is contained in a maximal totally real subalgebra, which generalizes [24, Lemma 4.1]. Consequently, we can apply Proposition 1.3 in order to prove Theorem 1.4 for \mathbb{B}_n .

Proposition 3.1. *Every totally real subalgebra \mathfrak{n}' of \mathfrak{n} is contained in a maximal totally real subalgebra $\widehat{\mathfrak{n}}'$ of \mathfrak{n} . Moreover, if \mathfrak{n}' is Abelian, then $\widehat{\mathfrak{n}}'$ can also be chosen to be Abelian.*

Proof. Let $\Phi: \mathfrak{n} \rightarrow \mathfrak{n} \cdot p_0$ be the linear isomorphism given by $\Phi(x) := x(p_0)$. Let $V \subset \mathfrak{n} \cdot p_0$ be a real vector subspace. The following is a direct consequence of equation (3.1).

- (a) The preimage $\Phi^{-1}(V)$ is an Abelian subalgebra of \mathfrak{n} if and only if V is isotropic with respect to ω_{p_0} .
- (b) The preimage $\Phi^{-1}(V)$ is a nonabelian subalgebra of \mathfrak{n} if and only if V is not isotropic and contains $\zeta(p_0)$.

Now firstly suppose that \mathfrak{n}' is Abelian. Then $\mathfrak{n}' \cdot p_0$ is not only totally real but isotropic, and by basic symplectic linear algebra there exists a Lagrangian subspace V of $\mathfrak{n} \cdot p_0$ that contains $\mathfrak{n}' \cdot p_0$. As we have noted above, the preimage $\widehat{\mathfrak{n}}' := \Phi^{-1}(V)$ is an Abelian subalgebra of maximal dimension of \mathfrak{n} that contains the totally real subalgebra \mathfrak{n}' .

In the case that \mathfrak{n}' is not Abelian, we choose any maximally totally real subspace V of $\mathfrak{n} \cdot p_0$ that contains $\mathfrak{n}' \cdot p_0$. Since $\mathfrak{n}' \cdot p_0$ contains $\zeta(p_0)$, the same holds for V . Hence, it follows that $\widehat{\mathfrak{n}}' := \Phi^{-1}(V)$ has the required properties. \square

4. THE CASE OF THE LIE BALL

In this section we first describe the structure of the normal j -algebra \mathfrak{l}_n of the Lie ball \mathbb{L}_n in order to establish in particular the existence of a holomorphic submersion from \mathbb{L}_n onto the unit disk \mathbb{B}_1 whose fibers are biholomorphic to \mathbb{B}_{n-1} . It turns out that this submersion is crucial for the proof of Theorem 1.4 since the analogon of Proposition 3.1 does *not* hold true for \mathfrak{l}_n .

4.1. The normal j -algebra of the Lie ball. The n -dimensional Lie ball is biholomorphically equivalent to the tube domain over the symmetric cone

$$\Omega := \{y \in \mathbb{R}^n; y_n^2 - (y_1^2 + \cdots + y_{n-1}^2) > 0, y_n > 0\}.$$

For $n = 1$ this is just the upper half-plane \mathbb{H}^+ , while for $n = 2$ we get $\mathbb{H}^+ \times \mathbb{H}^+$. Therefore, we will concentrate on the case $n \geq 3$.

In the notation of [16, Table V, Chapter X] the Lie ball \mathbb{L}_n corresponds to the item *BD I* with $p = 2$ and $q = n$. The connected component of its automorphism group is isomorphic to $G = \mathrm{SO}^0(2, n)$ and the subgroup of affine automorphisms is $G(\Omega) \times \mathbb{R}^n$ with $G(\Omega) = \mathbb{R}^{>0}\mathrm{SO}^0(1, n-1)$, see [10, Chapter X.5].

Remark. Since the group $G(\Omega)$ acts by matrix multiplication on \mathbb{C}^n , the vector field corresponding to an element x of the Lie algebra of $G(\Omega)$ can be obtained by computing $\left. \frac{d}{dt} \right|_0 \exp(tx)z$ for $z \in \mathbb{C}^n$.

In view of [20, Proposition 2.8], in order to describe the normal j -algebra $\mathfrak{l}_n = \mathfrak{a} \oplus \mathfrak{n}$ of the Lie ball, it is sufficient to find a maximal triangular subalgebra of $\mathfrak{so}(1, n-1)$. By doing so, we can choose the basis of \mathfrak{n} given by the vector fields

$$\begin{aligned} \xi_k &= \frac{\partial}{\partial z_k} \quad (1 \leq k \leq n-2), \\ \xi'_k &= (z_n - z_{n-1}) \frac{\partial}{\partial z_k} + z_k \frac{\partial}{\partial z_{n-1}} + z_k \frac{\partial}{\partial z_n} \quad (1 \leq k \leq n-2), \\ \zeta &= \frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_n}, \\ \eta &= \frac{\partial}{\partial z_n} - \frac{\partial}{\partial z_{n-1}}. \end{aligned}$$

The only non-vanishing Lie brackets are

$$[\xi_k, \xi'_k] = \zeta \quad \text{and} \quad [\eta, \xi'_k] = 2\xi_k$$

for all $1 \leq k \leq n-2$. In particular, we see that $\mathbb{R}\zeta$ is the center of \mathfrak{n} and that the Abelian Lie algebra $\mathfrak{n}' = \bigoplus_{k=1}^{n-2} \mathbb{R}\xi_k \oplus \mathbb{R}\zeta$ is its derived algebra.

A basis of \mathfrak{a} is given by

$$\begin{aligned} \delta &:= z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n} \quad (\text{the Euler field}), \text{ and} \\ \alpha &:= z_n \frac{\partial}{\partial z_{n-1}} + z_{n-1} \frac{\partial}{\partial z_n}. \end{aligned}$$

The action of \mathfrak{a} on \mathfrak{n} is determined by

$$[\delta, \xi_k] = -\xi_k, \quad [\delta, \zeta] = -\zeta, \quad [\delta, \eta] = -\eta$$

for all $1 \leq k \leq n-2$ and by

$$[\alpha, \zeta'_k] = -\zeta'_k, \quad [\alpha, \zeta] = -\zeta, \quad [\alpha, \eta] = \eta$$

for all $1 \leq k \leq n-2$.

To obtain the complex structure $j: \mathfrak{l}_n \rightarrow \mathfrak{l}_n$, let us fix the base point $p_0 = ie_n$ in $\widehat{\mathbb{L}}_n = \mathbb{R}^n + i\Omega$. This yields

$$j(\zeta'_k) = \zeta'_k, \quad j(\zeta) = \alpha + \delta =: \alpha_1, \quad j(\eta) = \delta - \alpha =: \alpha_2$$

for all $1 \leq k \leq n-2$.

4.2. The equivariant fibration of \mathbb{L}_n . It follows directly from the bracket relations described above that the subalgebra

$$\mathfrak{b}_{n-1} := \mathbb{R}\alpha_1 \oplus \bigoplus_{k=1}^{n-2} \mathbb{R}\zeta_k \oplus \bigoplus_{k=1}^{n-2} \mathbb{R}\zeta'_k \oplus \mathbb{R}\zeta$$

is a j -invariant ideal in \mathfrak{l}_n isomorphic to the normal j -algebra of the $(n-1)$ -dimensional unit ball. Moreover, the quotient $\mathfrak{b}_1 \cong \mathfrak{l}_n / \mathfrak{b}_{n-1}$ is isomorphic to the j -subalgebra $\mathbb{R}\alpha_2 \oplus \mathbb{R}\eta$ of \mathfrak{l}_n . In other words, \mathfrak{l}_n decomposes into the semi-direct product $\mathfrak{l}_n = \mathfrak{b}_{n-1} \ltimes \mathfrak{b}_1$. Let

$$L_n \cong B_{n-1} \rtimes B_1$$

be the corresponding decomposition at the group level.

Geometrically, this corresponds to the L_n -equivariant holomorphic submersion $\pi: \widehat{\mathbb{L}}_n \rightarrow \mathbb{C}$ given by $\pi(z) = z_n - z_{n-1}$. In fact, an elementary argument shows that for $z \in \widehat{\mathbb{L}}_n$ we have $z_n - z_{n-1} \in \mathbb{H}^+$ and $z_n + z_{n-1} \in \mathbb{H}^+$. Moreover, for $a := z_n - z_{n-1} \in \mathbb{H}^+$ and $w := z_n + z_{n-1} \in \mathbb{H}^+$, we obtain

$$\pi^{-1}(a) \cong \left\{ (z_1, \dots, z_{n-2}, w) \in \mathbb{C}^{n-1} \mid \operatorname{Im}(w) > \frac{\operatorname{Im}(z_1)^2 + \dots + \operatorname{Im}(z_{n-2})^2}{\operatorname{Im}(a)} \right\},$$

which is a realization of the $(n-1)$ -dimensional unit ball as an unbounded tube domain. Hence, all π -fibers are isomorphic to \mathbb{B}_{n-1} .

Remark. Since the Lie ball \mathbb{L}_n is irreducible for $n \geq 3$, the holomorphic submersion π is *not* holomorphically locally trivial unless $n = 2$, as follows from [31].

4.3. Proof of Theorem 1.4 for the Lie ball. Let Γ be a discrete subgroup of $\operatorname{Aut}^0(\mathbb{L}_n)$ such that $\Gamma \subset N$. Let N_Γ be the Zariski closure of Γ in N and suppose that \mathfrak{n}_Γ is totally real, i.e., that $\mathfrak{n}_\Gamma \cap j(\mathfrak{n}_\Gamma) = \{0\}$.

We start by giving an example that shows that Proposition 3.1 does no longer hold true for the Lie ball.

Example. Let us consider the 6-dimensional normal j -algebra $\mathfrak{l}_3 = \mathfrak{a} \oplus \mathfrak{n}$ of the 3-dimensional Lie ball. We claim that the totally real subalgebra $\mathbb{R}\zeta'_1 \subset \mathfrak{n}$ is not contained in a maximal totally real subalgebra of \mathfrak{n} .

Since $\mathfrak{b}_2 \triangleleft \mathfrak{l}_3$ is 4-dimensional and not totally real, a maximal totally real subalgebra of \mathfrak{n} must contain an element $x = a\zeta_1 + b\zeta'_1 + c\zeta + d\eta$ with $a, b, c, d \in \mathbb{R}$ and $d \neq 0$. Thus it also must contain

$$[x, \zeta'_1] = a\zeta + 2d\zeta_1 \quad \text{and} \quad [[x, \zeta'_1], \zeta'_1] = 2d\zeta.$$

Therefore it contains ζ and $\zeta_1 = -j(\zeta'_1)$ as well, which is impossible.

The following proposition is weaker than Proposition 3.1 but will still allow us to prove Theorem 1.4 for the Lie ball. For its statement and proof we decompose the nilradical \mathfrak{n} of \mathfrak{l}_n as $\mathfrak{n} = \mathfrak{n}_{n-1} \oplus \mathbb{R}\eta$ where the ideal $\mathfrak{n}_{n-1} := \mathfrak{b}_{n-1} \cap \mathfrak{n}$ is isomorphic to the nilradical of \mathfrak{b}_{n-1} .

Proposition 4.1. *Let \mathfrak{n}' be a totally real subalgebra of the nilradical $\mathfrak{n} = \mathfrak{n}_{n-1} \oplus \mathbb{R}\eta$. If \mathfrak{n}' is not contained in \mathfrak{n}_{n-1} , then there exists a maximal totally real subalgebra $\widehat{\mathfrak{n}}'$ of \mathfrak{n} which contains \mathfrak{n}' .*

Proof. Since by assumption \mathfrak{n}' is not contained in \mathfrak{n}_{n-1} , there is an element of the form

$$x_0 = \sum a_k \zeta_k + \sum b_l \zeta'_l + c\zeta + \eta$$

which belongs to \mathfrak{n}' . Moreover, we have $\dim(\mathfrak{n}' \cap \mathfrak{n}_{n-1}) = \dim \mathfrak{n}' - 1$.

As a first step, we claim that \mathfrak{n}' is totally real if and only if $\mathfrak{n}' \cap \mathfrak{n}_{n-1}$ is so. In order to see this, suppose that $\mathfrak{n}' \cap \mathfrak{n}_{n-1}$ is totally real, the other implication being trivial. Choose an element $x \in \mathfrak{n}' \cap j(\mathfrak{n}')$ and decompose it as $x = x_1 + tx_0$ where $x_1 \in \mathfrak{n}' \cap \mathfrak{n}_{n-1}$ and $t \in \mathbb{R}$. By assumption, we have $j(x) = j(x_1) + tj(x_0) \in \mathfrak{n}'$. Then, from $j(\eta) = \alpha_2$ we deduce $t = 0$, hence $x = x_1 \in \mathfrak{n}' \cap \mathfrak{n}_{n-1}$. Since the latter algebra is assumed to be totally real we obtain $x = 0$, as was to be shown.

Next, observe that if $\zeta \notin \mathfrak{n}'$, then $\mathfrak{n}' \oplus \mathbb{R}\zeta$ is again a totally real subalgebra, for the following reason. If $\zeta \notin \mathfrak{n}'$, then $\mathfrak{n}' \cap \mathfrak{n}_{n-1}$ is Abelian. Therefore, the Lie algebra $(\mathfrak{n}' \cap \mathfrak{n}_{n-1}) \oplus \mathbb{R}\zeta$ is also Abelian, hence totally real, which due to the previous step implies that $\mathfrak{n}' \oplus \mathbb{R}\zeta$ is totally real.

Consequently, we may assume that $\zeta \in \mathfrak{n}'$. Let us consider the projection $\pi: \mathfrak{n}_{n-1} \rightarrow \mathfrak{n}_{n-1}$ onto $\bigoplus_k \mathbb{R}\zeta_k \oplus \mathbb{R}\zeta$ with kernel $\bigoplus_l \mathbb{R}\zeta'_l$. If the restriction

$$\pi|_{\mathfrak{n}' \cap \mathfrak{n}_{n-1}}: \mathfrak{n}' \cap \mathfrak{n}_{n-1} \rightarrow \bigoplus_k \mathbb{R}\zeta_k \oplus \mathbb{R}\zeta$$

is surjective, then $\dim(\mathfrak{n}' \cap \mathfrak{n}_{n-1}) \geq n - 1$, hence $\dim \mathfrak{n}' = n$, and we are done.

Therefore, suppose that $\zeta_{k_0} \notin \pi(\mathfrak{n}' \cap \mathfrak{n}_{n-1})$ for some k_0 . Note that in particular $\zeta_{k_0} \notin \mathfrak{n}'$. From $\zeta \in \mathfrak{n}'$ it follows that $(\mathfrak{n}' \cap \mathfrak{n}_{n-1}) \oplus \mathbb{R}\zeta_{k_0}$ is a subalgebra of $\mathfrak{n}' \oplus \mathfrak{n}_{n-1}$. Since moreover, η and ζ_{k_0} commute, we see that $\mathfrak{n}' \oplus \mathbb{R}\zeta_{k_0}$ is a subalgebra of \mathfrak{n} . We claim that $\mathfrak{n}' \oplus \mathbb{R}\zeta_{k_0}$ is totally real. In order to see this it is sufficient to show that $(\mathfrak{n}' \cap \mathfrak{n}_{n-1}) \oplus \mathbb{R}\zeta_{k_0}$ is totally real. This, however, follows from the fact that $[x_0, j(\zeta_{k_0})]$ has a non-zero contribution from ζ_{k_0} .

Iterating this procedure we eventually construct a maximal totally real subalgebra of \mathfrak{n} that contains \mathfrak{n}' , as was to be shown. \square

Proof of Theorem 1.4 for the Lie ball. We shall prove that \mathbb{L}_n/Γ is Stein where Γ satisfies the conditions stated at the beginning of this subsection.

Let us first assume that \mathfrak{n}_Γ is not contained in the j -ideal \mathfrak{b}_{n-1} . Due to Proposition 4.1 there exists a maximal totally real subalgebra $\widehat{\mathfrak{n}}_\Gamma \subset \mathfrak{n}$ that contains \mathfrak{n}_Γ . Then the result follows from Proposition 1.3.

If \mathfrak{n}_Γ is contained in \mathfrak{b}_{n-1} , we find a maximal totally real subalgebra $\widehat{\mathfrak{n}}_\Gamma$ of \mathfrak{b}_{n-1} that contains \mathfrak{n}_Γ , see Proposition 3.1. Let \widehat{N}_Γ be the corresponding closed subgroup of $\text{Aut}^0(\mathbb{L}_n)$. The holomorphic submersion $\pi: \widehat{\mathbb{L}}_n \rightarrow \mathbb{H}^+$ is \widehat{N}_Γ -invariant and extends to a $\widehat{N}_\Gamma^{\mathbb{C}}$ -invariant holomorphic submersion $\widehat{\pi}: \widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{\mathbb{L}}_n \rightarrow \mathbb{H}^+$. Then, $\widehat{N}_\Gamma^{\mathbb{C}}$ acts freely and properly on $\widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{\mathbb{L}}_n$, and the fibers of $\widehat{\pi}$ are precisely the $\widehat{N}_\Gamma^{\mathbb{C}}$ -orbits, isomorphic to \mathbb{C}^{n-1} . In other words, $\widehat{\pi}$ defines an $\widehat{N}_\Gamma^{\mathbb{C}}$ -principal bundle. Since \mathbb{H}^+ is contractible, this $\widehat{N}_\Gamma^{\mathbb{C}}$ -principal bundle is holomorphically trivial, i.e., we have

$$(\widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{\mathbb{L}}_n)/\Gamma \cong (\widehat{N}_\Gamma^{\mathbb{C}}/\Gamma) \times \mathbb{H}^+.$$

Since $\widehat{N}_\Gamma^{\mathbb{C}}/\Gamma$ is Stein due to [23, Théorème 1], the same is true for $(\widehat{N}_\Gamma^{\mathbb{C}} \cdot \widehat{\mathbb{L}}_n)/\Gamma$. Hence, the theorem of Docquier-Grauert [8] implies that $\widehat{\mathbb{L}}_n/\Gamma$ is Stein as well. \square

5. THE CASE OF THE SIEGEL DISK

In this section we present an example that shows that the analoga of Theorem 1.4 and Proposition 4.1 do not hold true for arbitrary bounded homogeneous domains. This example will be constructed on a 5-dimensional bounded homogeneous domain holomorphically embedded in the 6-dimensional Siegel disk S_3 . Therefore we describe as a first step the normal j -algebra \mathfrak{s}_n of S_n .

5.1. The normal j -algebra of Siegel's upper half-plane. Recall that the Siegel disk S_n can be realized as Siegel's upper half plane which is the symmetric tube domain associated with the cone of positive definite real symmetric matrices. The automorphism group of S_n is isomorphic to the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ and S_n corresponds to the hermitian symmetric space of type C I in [16]. For $n = 1$, Siegel's upper half-plane is the usual upper half-plane, while, for $n = 2$, it is isomorphic to the 3-dimensional Lie ball.

The linear automorphism group of the symmetric cone of positive definite real symmetric matrices is $\mathrm{GL}(n, \mathbb{R})$ acting by $g \cdot A := gAg^t$, see [10, p. 213]. Hence, the normal j -algebra \mathfrak{s}_n of S_n can be determined as before using [10, Proposition X.5.4] and [20, Proposition 2.8]. Our description of \mathfrak{s}_n follows [18].

Let us denote by \mathfrak{u}_n the solvable Lie algebra of lower triangular real $n \times n$ matrices. We have the decomposition $\mathfrak{u}_n = \mathfrak{a}_n \oplus \mathfrak{u}'_n$ where \mathfrak{a}_n is the Abelian Lie algebra of diagonal matrices in $\mathbb{R}^{n \times n}$. The normal j -algebra of Siegel's upper half plane can be realized as

$$\mathfrak{s}_n := \left\{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \mid A \in \mathfrak{u}_n, B \in \mathrm{Sym}(n, \mathbb{R}) \right\}.$$

Note that $\varphi: \mathfrak{u}_n \rightarrow \mathrm{Sym}(n, \mathbb{R})$, $\varphi(A) = A + A^t$, is a linear isomorphism. The complex structure $j: \mathfrak{s}_n \rightarrow \mathfrak{s}_n$ is given by

$$j \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} = \begin{pmatrix} \varphi^{-1}(B) & -\varphi(A) \\ 0 & -\varphi^{-1}(B)^t \end{pmatrix}.$$

The linear form $\lambda \in \mathfrak{s}_n^*$ corresponding to the Bergman metric is given by

$$\lambda \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} = \mathrm{Tr}(B).$$

The elements of the solvable Lie group S_n are of the form $\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix}$ where A is lower triangular and B is symmetric. An element of the form $\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$ acts on S_n by $Z \mapsto AZA^t$ while the elements of the form $\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$ act by translation. The vector field induced by $x \in \mathfrak{s}_n$ is given by $\left. \frac{d}{dt} \right|_0 \exp(tx) \cdot Z$ for $Z \in S_n$.

One verifies directly that the subspace of \mathfrak{s}_n consisting of all matrices $A = (a_{kl})$ and $B = (b_{kl})$ such that $a_{kl} = b_{kl} = 0$ for all $1 \leq k, l \leq n-1$ is a j -invariant ideal isomorphic to \mathfrak{b}_n , while the subspace consisting of all matrices where $a_{nl} = b_{nl} = 0$ for all l is a j -invariant complementary subalgebra isomorphic to \mathfrak{s}_{n-1} . Consequently, \mathfrak{s}_n decomposes as a semi-direct sum $\mathfrak{s}_{n-1} \oplus \mathfrak{b}_n$. Geometrically, this decomposition corresponds to the equivariant fibration $\pi_n: \widehat{S}_n \rightarrow \widehat{S}_{n-1}$ where $\pi_n(Z)$ is the submatrix of Z consisting of the first $n-1$ lines and columns.

In the following we will concentrate on the case $n = 3$. Here we have $\mathfrak{s}_3 = \mathfrak{b}_3 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_1$ where \mathfrak{b}_3 is a j -ideal in \mathfrak{s}_3 and \mathfrak{b}_2 is a j -ideal in $\mathfrak{s}_2 = \mathfrak{b}_2 \oplus \mathfrak{b}_1$. The composition of the two equivariant fibrations is the map $\pi := \pi_2 \circ \pi_3: \widehat{S}_3 \rightarrow \mathbb{H}^+$ given by $\pi(Z) = z_{11}$. For

the rest of this section let $D := \pi^{-1}(i)$ be the 5-dimensional bounded homogeneous (non-symmetric) domain corresponding to the normal j -algebra $\mathfrak{b}_3 \oplus \mathfrak{b}_2$. Let us consider the bases $(\alpha_3, \zeta_{31}, \zeta_{32}, \zeta'_{31}, \zeta'_{32}, \zeta_3)$ of \mathfrak{b}_3 and $(\alpha_2, \zeta_{21}, \zeta'_{21}, \zeta_2)$ of \mathfrak{b}_2 . Let us realize these elements explicitly as matrices as well as vector fields in the coordinates of the entries of $Z = (z_{kl}) \in \text{Sym}(3, \mathbb{C})$:

$$\begin{aligned} \zeta_3 &= -2 \begin{pmatrix} 0 & E_{33} \\ 0 & 0 \end{pmatrix} \mapsto -2 \frac{\partial}{\partial z_{33}}, & \alpha_3 &= \begin{pmatrix} E_{33} & 0 \\ 0 & -E_{33} \end{pmatrix} \mapsto z_{13} \frac{\partial}{\partial z_{13}} + z_{23} \frac{\partial}{\partial z_{23}} + 2z_{33} \frac{\partial}{\partial z_{33}}, \\ \zeta_{31} &= \begin{pmatrix} 0 & E_{13} + E_{31} \\ 0 & 0 \end{pmatrix} \mapsto \frac{\partial}{\partial z_{13}}, & \zeta'_{31} &= \begin{pmatrix} E_{31} & 0 \\ 0 & -E_{13} \end{pmatrix} \mapsto z_{11} \frac{\partial}{\partial z_{13}} + z_{12} \frac{\partial}{\partial z_{23}} + 2z_{13} \frac{\partial}{\partial z_{33}}, \\ \zeta_{32} &= \begin{pmatrix} 0 & E_{23} + E_{32} \\ 0 & 0 \end{pmatrix} \mapsto \frac{\partial}{\partial z_{23}}, & \zeta'_{32} &= \begin{pmatrix} E_{32} & 0 \\ 0 & -E_{23} \end{pmatrix} \mapsto z_{12} \frac{\partial}{\partial z_{13}} + z_{22} \frac{\partial}{\partial z_{23}} + 2z_{23} \frac{\partial}{\partial z_{33}}, \\ \zeta_2 &= -2 \begin{pmatrix} 0 & E_{22} \\ 0 & 0 \end{pmatrix} \mapsto -2 \frac{\partial}{\partial z_{22}}, & \alpha_2 &= \begin{pmatrix} E_{22} & 0 \\ 0 & -E_{22} \end{pmatrix} \mapsto z_{12} \frac{\partial}{\partial z_{12}} + 2z_{22} \frac{\partial}{\partial z_{22}} + z_{23} \frac{\partial}{\partial z_{23}}, \\ \zeta_{21} &= \begin{pmatrix} 0 & E_{12} + E_{21} \\ 0 & 0 \end{pmatrix} \mapsto \frac{\partial}{\partial z_{12}}, & \zeta'_{21} &= \begin{pmatrix} E_{21} & 0 \\ 0 & -E_{12} \end{pmatrix} \mapsto z_{11} \frac{\partial}{\partial z_{12}} + 2z_{12} \frac{\partial}{\partial z_{22}} + z_{13} \frac{\partial}{\partial z_{23}}. \end{aligned}$$

For instance, in order to find the vector field belonging to ζ'_{31} we calculate

$$\exp(t\zeta'_{31}) \cdot Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{pmatrix} Z \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & z_{13} + tz_{11} \\ z_{12} & z_{22} & z_{23} + tz_{12} \\ z_{13} + tz_{11} & z_{23} + tz_{12} & z_{33} + 2tz_{13} + t^2z_{11} \end{pmatrix}$$

and then derive with respect to t .

The representation of \mathfrak{b}_2 on \mathfrak{b}_3 is defined by

$$\begin{aligned} [\alpha_2, \zeta_{32}] &= \zeta_{32} & [\alpha_2, \zeta'_{32}] &= -\zeta'_{32} \\ [\zeta_{21}, \zeta'_{31}] &= -\zeta_{32} & [\zeta_{21}, \zeta_{32}] &= -\zeta_{31} \\ [\zeta'_{21}, \zeta_{31}] &= \zeta_{32} & [\zeta'_{21}, \zeta'_{32}] &= -\zeta'_{31} \\ [\zeta_2, \zeta'_{32}] &= 2\zeta_{32}, \end{aligned}$$

all other brackets being zero.

5.2. A counterexample. We present an example that shows that the analoga of Proposition 4.1 and Theorem 1.4 do not hold true for D .

Let \mathfrak{n} denote the nilradical of $\mathfrak{b}_3 \oplus \mathfrak{b}_2$ and consider the elements $x_1, x_2, x_3 \in \mathfrak{n}$ given by

$$x_1 := \zeta'_{31} + \zeta_3 + \zeta_{21}, \quad x_2 := -\zeta_{31} + \zeta'_{21}, \quad x_3 := \zeta_3 + \zeta_2.$$

Since $[x_1, x_2] = x_3$, they generate a Lie subalgebra $\mathfrak{n}_\Gamma := \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \mathbb{R}x_3$ that is isomorphic to the 3-dimensional Heisenberg algebra and projects surjectively onto the nilradical of \mathfrak{b}_2 . Moreover, the corresponding group N_Γ of automorphisms of D admits a cocompact discrete subgroup, which justifies the notation \mathfrak{n}_Γ .

We shall see that all N_Γ -orbits in D are totally real, while \mathfrak{n}_Γ is not contained in any maximal totally real subalgebra of \mathfrak{n} . This shows that the analogon of Proposition 4.1 does not hold true for the domain D , even under the stronger assumption that *all* N_Γ -orbits are totally real in D .

Remark. A subalgebra of codimension 1 in a nilpotent Lie algebra is automatically an ideal. We apply this result in the following way: If $y \in \mathfrak{n}$ is an element such that $\mathfrak{n}_\Gamma \oplus \mathbb{R}y$ is a subalgebra, then y normalizes \mathfrak{n}_Γ .

Lemma 5.1. *The Lie algebra \mathfrak{n}_Γ is not contained in a maximal totally real subalgebra of \mathfrak{n} . Hence, the analogon of Proposition 4.1 does not hold true for the domain D .*

Proof. Suppose for a moment that $\widehat{\mathfrak{n}}_\Gamma$ is a maximal totally real subalgebra of \mathfrak{n} with $\mathfrak{n}_\Gamma \subset \widehat{\mathfrak{n}}_\Gamma$ and let $y \in \widehat{\mathfrak{n}}_\Gamma \setminus \mathfrak{n}_\Gamma$. Writing

$$y = a_1\check{\zeta}_{31} + a_2\check{\zeta}_{32} + a'_1\check{\zeta}'_{31} + a'_2\check{\zeta}'_{32} + c_3\zeta_3 + b_1\check{\zeta}_{21} + b'_1\check{\zeta}'_{21} + c_2\zeta_2$$

we find

$$[y, x_1] = a'_2\check{\zeta}_{31} + (a'_1 - b_1)\check{\zeta}_{32} + a_1\zeta_3 - b'_1\zeta_2 \quad \text{and} \quad [[y, x_1], x_1] = a'_2\zeta_3.$$

If $a'_2 \neq 0$, then $\widehat{\mathfrak{n}}_\Gamma$ would contain ζ_3 and ζ_2 and consequently $\check{\zeta}'_{31} + \check{\zeta}_{21} = -j(x_2)$ as well, thus contradicting the fact that $\widehat{\mathfrak{n}}_\Gamma$ is totally real. Hence, we must have $a'_2 = 0$.

It follows that the elements

$$\begin{aligned} [y, x_1] &= (a'_1 - b_1)\check{\zeta}_{32} + a_1\zeta_3 - b'_1\zeta_2 \quad \text{and} \\ [y, x_2] &= -(a_1 + b'_1)\check{\zeta}_{32} + a'_1\zeta_3 + b_1\zeta_2 \end{aligned}$$

belong to $\widehat{\mathfrak{n}}_\Gamma$. Consequently, $\widehat{\mathfrak{n}}_\Gamma$ contains also the element

$$(a_1 + b'_1)[y, x_1] + (a'_1 - b_1)[y, x_2] = (a_1^2 - a'_1b_1 + a_1b'_1 + a_1'^2)\zeta_3 - (b_1^2 - a'_1b_1 + a_1b'_1 + b_1'^2)\zeta_2.$$

As above, since $\widehat{\mathfrak{n}}_\Gamma$ is totally real, neither ζ_3 nor ζ_2 belong to $\widehat{\mathfrak{n}}_\Gamma$. This gives

$$a_1^2 - a'_1b_1 + a_1b'_1 + a_1'^2 = -b_1^2 + a'_1b_1 - a_1b'_1 - b_1'^2,$$

which is equivalent to

$$(a_1 + b'_1)^2 + (a'_1 - b_1)^2 = 0.$$

Hence, we obtain $a_1 = -b'_1$ and $a'_1 = b_1$. In summary, this yields

$$y = a_1(\check{\zeta}_{31} - \check{\zeta}'_{21}) + a_2\check{\zeta}_{32} + a'_1(\check{\zeta}'_{31} + \check{\zeta}_{21}) + c_3\zeta_3 + c_2\zeta_2.$$

In particular y normalizes \mathfrak{n}_Γ .

Adding $a_1x_2 - a'_1x_1 - c_2x_3$ to y , we conclude that $\widehat{\mathfrak{n}}_\Gamma \setminus \mathfrak{n}_\Gamma$ contains an element of the form

$$(5.1) \quad y_\tau := \check{\zeta}_{32} + \tau\zeta_3$$

for some $\tau \in \mathbb{R}$. Observe that y_τ centralizes \mathfrak{n}_Γ and that $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$ is a totally real subalgebra of \mathfrak{n} .

According to the above remark, $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$ must be a normal subalgebra of $\widehat{\mathfrak{n}}_\Gamma$. Therefore, we calculate its normalizer and find

$$\mathcal{N}_{\mathfrak{n}}(\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau) = \mathfrak{n}_\Gamma \oplus \mathbb{R}\check{\zeta}_{32} \oplus \mathbb{R}\zeta_3 = \mathbb{R}(\check{\zeta}'_{31} + \check{\zeta}_{21}) \oplus \mathbb{R}(-\check{\zeta}_{31} + \check{\zeta}'_{21}) \oplus \mathbb{R}\check{\zeta}_{32} \oplus \mathbb{R}\zeta_3 \oplus \mathbb{R}\zeta_2.$$

Since the normalizer is 5-dimensional, it must coincide with $\widehat{\mathfrak{n}}_\Gamma$. This however contradicts our assumption because the normalizer is *not* totally real. \square

Let \widehat{N}_Γ be the connected Lie group having Lie algebra $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$ where y_τ is defined in Equation (5.1). Since y_τ centralizes \mathfrak{n}_Γ , the group \widehat{N}_Γ admits cocompact discrete subgroups. We will show that the quotient of D with respect to any cocompact discrete subgroup of \widehat{N}_Γ is *not* Stein, hence that the analogon of Theorem 1.4 does not hold for D . For this it is enough to find an \widehat{N}_Γ -orbit in D which is not totally real. Note that this implies again that $\mathfrak{n}_\Gamma \oplus \mathbb{R}y_\tau$ cannot be contained in a maximal totally real subalgebra. Indeed, for every $\alpha > (1 + \tau^2)^{-2}$ the matrix

$$Z_0 = \begin{pmatrix} i & \frac{\tau+i}{1+\tau^2} & 0 \\ \frac{\tau+i}{1+\tau^2} & i\alpha & 0 \\ 0 & 0 & i \end{pmatrix}$$

belongs to D , and one sees without difficulty that $\widehat{N}_\Gamma \cdot Z_0$ is not totally real.

Nevertheless, we shall prove in the rest of this subsection that D/Γ is a Stein manifold where Γ is any cocompact discrete subgroup of N_Γ . Using the description of the elements of \mathfrak{s}_n as matrices, we can realize $N_\Gamma^\mathbb{C}$ as the matrix group

$$(5.2) \quad N_\Gamma^\mathbb{C} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & a & -b \\ b & 1 & 0 & a & -2c & -\frac{a^2+b^2}{2} \\ a & 0 & 1 & -b & \frac{a^2+b^2}{2} & -2(a+c) \\ 0 & 0 & 0 & 1 & -b & -a \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}.$$

The action of $N_\Gamma^\mathbb{C}$ on $\text{Sym}(3, \mathbb{C})$ is given by

$$Z = (z_{kl}) \mapsto \begin{pmatrix} z_{11} & z_{12} + bz_{11} + a & z_{13} + az_{11} - b \\ z_{12} + bz_{11} + a & z_{22} + 2bz_{12} + b^2z_{11} + ab - 2c & z_{23} + az_{12} + bz_{13} + abz_{11} + \frac{a^2-b^2}{2} \\ z_{13} + az_{11} - b & z_{23} + az_{12} + bz_{13} + abz_{11} + \frac{a^2-b^2}{2} & z_{33} + 2az_{13} + a^2z_{11} - ab - 2(a+c) \end{pmatrix}.$$

It is not difficult to verify that $N_\Gamma^\mathbb{C}$ acts freely on $\text{Sym}(3, \mathbb{C})$, which in particular implies that N_Γ has only totally real orbits in D , see Lemma 2.1.

In the following we will show that $N_\Gamma^\mathbb{C}$ acts properly on $\text{Sym}(3, \mathbb{C})$ with quotient manifold $\text{Sym}(3, \mathbb{C})/N_\Gamma^\mathbb{C} \cong \mathbb{C}^3$. It then follows that the principal bundle $\text{Sym}(3, \mathbb{C}) \rightarrow \mathbb{C}^3$ is holomorphically trivial. Consequently, $\text{Sym}(3, \mathbb{C})/\Gamma$ and D/Γ are Stein.

In order to prove that the $N_\Gamma^\mathbb{C}$ -action is proper we shall use the following lemma.

Lemma 5.2. *Let G be a Lie group acting smoothly and freely on a manifold M and let H be a closed normal subgroup of G . Then G acts properly on M if and only if H acts properly on M and G/H acts properly on M/H .*

Proof. Firstly, suppose that G acts properly on M . Since H is closed in G , the H -action on M is proper and we get a smooth action of G/H on the quotient manifold M/H . Properness of this latter action was shown in [29, Proposition 1.3.2].

Conversely, suppose that the actions of H on M and of G/H on M/H are proper. Let (g_n) and (p_n) be sequences in G and M , respectively, such that $(g_n \cdot p_n, p_n)$ converges to (q_0, p_0) . We must show that (g_n) has a convergent subsequence.

For this, note that $(g_n H \cdot [p_n], [p_n])$ converges to $([q_0], [p_0])$ in $M/H \times M/H$ where $[p] \in M/H$ denotes the class of p modulo H . Thus $(g_n H)$ has a convergent subsequence. Without loss of generality we assume that $g_n H \rightarrow g_0 H$. Hence, there is a sequence (h_n) in H such that $h_n^{-1} g_n \rightarrow g_0$ in G , from which we conclude that

$$(h_n \cdot (h_n^{-1} g_n \cdot p_n), h_n^{-1} g_n \cdot p_n) \rightarrow (q_0, g_0 \cdot p_0).$$

Since H acts properly on M , it follows that (h_n) and hence (g_n) have convergent subsequences. \square

We first apply Lemma 5.2 to the action of the center of $N_\Gamma^\mathbb{C}$ on $\text{Sym}(3, \mathbb{C})$. The center of $N_\Gamma^\mathbb{C}$ is given by all matrices of the form (5.2) having $a = b = 0$ and acts on $\text{Sym}(3, \mathbb{C})$ by

$$c \cdot \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} - 2c & z_{23} \\ z_{13} & z_{23} & z_{33} - 2c \end{pmatrix}.$$

Hence, the corresponding \mathbb{C} -principal bundle is holomorphically trivial and given by

$$\pi: \text{Sym}(3, \mathbb{C}) \rightarrow \mathbb{C}^5, \quad \pi(z_{kl}) = (z_{11}, z_{12}, z_{13}, z_{22} - z_{33}, z_{23}).$$

The induced action of $\mathbb{C}^2 \cong N_\Gamma^\mathbb{C} / Z(N_\Gamma^\mathbb{C})$ on \mathbb{C}^5 is given by

$$(a, b) \cdot z = \begin{pmatrix} z_1 \\ z_2 + bz_1 + a \\ z_3 + az_1 - b \\ z_4 + 2bz_2 - 2az_3 + (b^2 - a^2)z_1 + 2ab + 2a \\ z_5 + az_2 + bz_3 + abz_1 + (a^2 - b^2)/2 \end{pmatrix}.$$

In the next step we consider only the action of the one parameter group \mathbb{C}_a of elements $(a, 0)$. A direct calculation shows that the \mathbb{C}_a -action on \mathbb{C}^5 becomes a translation in the first coordinate after conjugation by the biregular map $\Phi: \mathbb{C}^5 \rightarrow \mathbb{C} \times \mathbb{C}^4$,

$$z \mapsto (z_2, z_1, z_3 - z_1z_2, z_4 + 2z_2z_3 - 2z_1z_5 - 2z_2, z_2^2 - 2z_5).$$

It follows that \mathbb{C}_a acts properly on \mathbb{C}^5 and that the corresponding \mathbb{C}_a -principal bundle $\mathbb{C}^5 \rightarrow \mathbb{C}^5/\mathbb{C}_a \cong \mathbb{C}^4$ is holomorphically trivial. Moreover, the induced \mathbb{C}_b -action on \mathbb{C}^4 is of the form

$$b \cdot w = \begin{pmatrix} w_1 \\ w_2 - b(w_1^2 + 1) \\ w_3 - 2bw_1 \\ w_4 - 2bw_2 + b^2(w_1^2 + 1) \end{pmatrix}.$$

Note that the projection onto the first three coordinates is equivariant; therefore we obtain a free algebraic \mathbb{C} -action on \mathbb{C}^3 of the form $t \cdot w = (w_1, w_2 + tf(w_1), w_3 + tg(w_1))$ where $f(w_1) = -(w_1^2 + 1)$ and $g(w_1) = -2w_1$. Due to a result of Kaliman, see [19], every free algebraic \mathbb{C} -action on \mathbb{C}^3 is conjugate under a biregular map to a translation. Hence, it follows that \mathbb{C}_b acts properly on \mathbb{C}^3 and thus on \mathbb{C}^4 . Moreover, we have $\mathbb{C}^4/\mathbb{C}_b \cong \mathbb{C}^3$. Consequently, Lemma 5.2 applies to show that $N_\Gamma^\mathbb{C}$ acts properly on $\text{Sym}(3, \mathbb{C})$ with quotient \mathbb{C}^3 .

Remark. In our case it is possible to give an elementary proof of Kaliman's result which applies in a slightly more general setup. For this, let $f, g \in \mathcal{O}(\mathbb{C})$ be two entire functions and consider the holomorphic \mathbb{C} -action on \mathbb{C}^3 given by $t \cdot z = (z_1, z_2 + tf(z_1), z_3 + tg(z_1))$. This action is free if and only if f and g have no common zeros. In this case, there exist entire functions $\varphi, \psi \in \mathcal{O}(\mathbb{C})$ such that $\varphi f + \psi g = 1$, see [17]. Then the biholomorphic map $\Phi: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ given by

$$\Phi(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(z_1) & g(z_1) \\ 0 & -\psi(z_1) & \varphi(z_1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

transforms the \mathbb{C} -action into a translation.

5.3. Open problems and concluding remarks. Let $\Gamma \subset N$ be a discrete subgroup with Zariski closure N_Γ . As we have seen, in general the condition that only one N_Γ -orbit is totally real in D does not imply that D/Γ is Stein. It is natural to ask, however, whether D/Γ is Stein if all N_Γ -orbits in D are totally real. In this case, we obtain a free algebraic action of $N_\Gamma^\mathbb{C}$ on the domain $N_\Gamma^\mathbb{C} \cdot \widehat{D} \subset \mathbb{C}^n$.

As we have observed in the above examples, even if all N_Γ -orbits are totally real in D , the hypothesis of Proposition 1.3 does not need to hold true. Then, in order to answer the above question, one could try a direct approach similar to the one carried out in the previous subsection. This, however, poses two major problems. First of all, one has to

show that $N_{\Gamma}^{\mathbb{C}}$ acts properly on $N_{\Gamma}^{\mathbb{C}} \cdot \widehat{D} \subset \mathbb{C}^n$. Secondly, if $N_{\Gamma}^{\mathbb{C}}$ does indeed act properly, one must prove that $(N_{\Gamma}^{\mathbb{C}} \cdot \widehat{D}) / N_{\Gamma}^{\mathbb{C}}$ is a Stein manifold. These questions are far from being trivial, bearing in mind that there is an example of a proper algebraic \mathbb{C}^2 -action on \mathbb{C}^6 by affine-linear transformations such that $\mathbb{C}^6 / \mathbb{C}^2$ is quasi-affine but not Stein, see [36].

REFERENCES

- [1] D. Achab, F. Betten, and B. Krötz, *Discrete group actions on Stein domains in complex Lie groups*, Forum Math. **16** (2004), no. 1, 37–68.
- [2] A. Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.
- [3] D. Burns, S. Halverscheid, and R. Hind, *The geometry of Grauert tubes and complexification of symmetric spaces*, Duke Math. J. **118** (2003), no. 3, 465–491.
- [4] D. Burns, Jr. and S. Shnider, *Spherical hypersurfaces in complex manifolds*, Invent. Math. **33** (1976), no. 3, 223–246.
- [5] H. Cartan, *Sur les groupes de transformations analytiques*, (Exposés mathématiques IX.) Actual. scient. et industr., vol. 198, Hermann, Paris, 1935.
- [6] B.-Y. Chen, *Discrete groups and holomorphic functions*, Math. Ann. **355** (2013), no. 3, 1025–1047.
- [7] S. Dey and M. Kapovich, *A note on complex-hyperbolic Kleinian groups*, Arnold Math. J. **6** (2020), no. 3-4, 397–406.
- [8] F. Docquier and H. Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123.
- [9] C. de Fabritiis and A. Iannuzzi, *Quotients of the unit ball of \mathbb{C}^n for a free action of \mathbb{Z}* , J. Anal. Math. **85** (2001), 213–224.
- [10] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1994.
- [11] B. Gilligan and A. T. Huckleberry, *On non-compact complex nil-manifolds*, Math. Ann. **238** (1978), no. 1, 39–49.
- [12] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.
- [13] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.
- [14] F. Reese Harvey and R. O. Wells, Jr., *Zero sets of non-negative strictly plurisubharmonic functions*, Math. Ann. **201** (1973), 165–170.
- [15] P. Heinzner, *Geometric invariant theory on Stein spaces*, Math. Ann. **289** (1991), no. 4, 631–662.
- [16] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, corrected reprint of the 1978 original, Graduate Studies in Mathematics **34**, American Mathematical Society, Providence, RI, 2001.
- [17] O. Helmer, *Divisibility properties of integral functions*, Duke Math. J. **6** (1940), 345–356.
- [18] H. Ishi, *On symplectic representations of normal \mathfrak{j} -algebras and their application to Xu’s realizations of Siegel domains*, Differential Geom. Appl. **24** (2006), no. 6, 588–612.
- [19] S. Kaliman, *Free \mathbb{C}_+ -actions on \mathbb{C}^3 are translations*, Invent. Math. **156** (2004), no. 1, 163–173.
- [20] S. Kaneyuki, *Homogeneous bounded domains and Siegel domains*, Lecture Notes in Mathematics **241**, Springer-Verlag, Berlin-New York, 1971.
- [21] S. Kobayashi, *Hyperbolic complex spaces*, Grundlehren der mathematischen Wissenschaften **318**, Springer-Verlag, Berlin, 1998.
- [22] J. L. Koszul, *Sur la forme hermitienne canonique des espaces homogènes complexes*, Canadian J. Math. **7** (1955), 562–576.
- [23] J.-J. Loeb, *Action d’une forme réelle d’un groupe de Lie complexe sur les fonctions plurisousharmoniques*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 59–97.
- [24] C. Miebach, *Quotients of bounded homogeneous domains by cyclic groups*, Osaka J. Math. **47** (2010), no. 2, 331–352.
- [25] C. Miebach, *Sur les quotients discrets de semi-groupes complexes*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 2, 269–276.
- [26] C. Miebach and K. Oeljeklaus, *On proper \mathbb{R} -actions on hyperbolic Stein surfaces*, Doc. Math. **14** (2009), 673–689.
- [27] C. Miebach and K. Oeljeklaus, *Schottky group actions in complex geometry*, Geometric complex analysis, Springer Proc. Math. Stat., vol. 246, Springer, Singapore, 2018, pp. 257–268.
- [28] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. **22** (1957), p. iii+123.
- [29] R. S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323.

- [30] I. I. Pyateskii-Shapiro, *Automorphic functions and the geometry of classical domains*, Translated from the Russian. Mathematics and Its Applications, Vol. 8, Gordon and Breach Science Publishers, New York, 1969.
- [31] H. L. Royden, *Holomorphic fiber bundles with hyperbolic fiber*, Proc. Amer. Math. Soc. **43** (1974), 311–312.
- [32] È. B. Vinberg, *The Morozov-Borel theorem for real Lie groups*, Dokl. Akad. Nauk SSSR **141** (1961), 270–273.
- [33] È. B. Vinberg (ed.), *Lie groups and Lie algebras, III*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994.
- [34] È. B. Vinberg, S. G. Gindikin, and I. I. Pyateskii-Shapiro, *Classification and canonical realization of complex homogeneous bounded domains*, Trudy Moskov. Mat. Obšč. **12** (1963), 359–388.
- [35] S. Vitali, *Quotients of the crown domain by a proper action of a cyclic group*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 3227–3239.
- [36] J. Winkelmann, *On free holomorphic \mathbf{C} -actions on \mathbf{C}^n and homogeneous Stein manifolds*, Math. Ann. **286** (1990), no. 1-3, 593–612.

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