

Hodge structure of O’Grady’s Singular Moduli Spaces

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ABSTRACT. We investigate the Hodge structure of the singular O’Grady’s six and ten dimensional examples of irreducible symplectic varieties. In particular, we compute some of their Betti numbers and their Euler characteristic. As consequence, we deduce that these varieties do not have finite quotient singularities answering a question of Bakker and Lehn.

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1. INTRODUCTION AND NOTATIONS

Let X be a K3 surface or an abelian surface and H an ample divisor on it. We fix the Mukai vector $v = (2, 0, -2) \in H^{2*}(X, \mathbb{Z})$ and let $M_v(X, H)$ be the associated moduli space of S -equivalence classes of semi-stable sheaves, where we assume that H is a v -generic polarization on X . If $X = S$ is a K3 surface then $M := M_v(S, H)$ is O’Grady’s singular moduli space of dimension 10, see [18]. If $X = A$ is an abelian surface we fix $[F_0] \in M_v(A, H)$, where we assume that $[F_0]$ is the S -equivalence class of the semistable sheaf F_0 , and consider the isotrivial fibration

$$\begin{aligned} \text{Alb} : M_v(A, H) &\rightarrow A \times A^\vee \\ [F] &\mapsto \left(\sum c_2(F), \det(F) \otimes \det(F_0)^{-1} \right) \end{aligned}$$

where the summation is with respect to the group structure of A . The fiber $K := \text{Alb}^{-1}(0_A, \mathcal{O}_A)$ is O’Grady’s singular moduli space of dimension 6, see [19].

Let \widetilde{M} and \widetilde{K} be the blow-up of M and K along their singular loci; Lehn and Sorger [14, Théorème 1.1] proved that \widetilde{M} and \widetilde{K} are smooth irreducible holomorphically

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symplectic varieties, i.e. simply connected compact Kähler manifolds such that the space of global holomorphic 2-forms is generated by a symplectic form. Furthermore, the manifolds \widetilde{M} and \widetilde{K} are the smooth 10 dimensional and 6 dimensional example by O'Grady [18], [19].

The Hodge structure of the manifolds \widetilde{M} and \widetilde{K} has been recently computed by De Cataldo, Rapagnetta, Saccà [5] and Mongardi, Rapagnetta, Saccà [17] respectively. The purpose of this paper is to study the rational cohomology groups of the singular varieties K and M , and in particular their Hodge structures.

Theorem 1.1. *Let $\pi: \widetilde{M} \rightarrow M$ be the symplectic resolution of singularities mentioned above.*

- *The pullback*

$$\pi^*: H^{2k}(M, \mathbb{Q}) \rightarrow H^{2k}(\widetilde{M}, \mathbb{Q})$$

is injective for any k .

- *The cohomology groups*

$$H^{2k}(M, \mathbb{Q}) \text{ and } H^{2k+1}(M, \mathbb{Q})$$

carry a pure Hodge structure of weight $2k$.

- *We have the following Betti numbers of M :*

	b_0	b_1	b_2	b_3	b_4	b_5	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}
M	1	0	23	0	276	0	277	0	23	0	1

- *The Euler characteristic of M is $\chi(M) = 123606$.*

Even though we do not determine all Betti numbers, we compute some bounds on all of them, cfr. Corollaries 5.9, 5.13. Regarding the second point of the theorem above, observe that we do not exclude that all the cohomology groups $H^{2k+1}(M, \mathbb{Q})$ vanish and hence that all these Hodge structures are trivial. In fact, the odd Betti numbers we have determined so far are all zero.

Theorem 1.2. *Let $\pi: \widetilde{K} \rightarrow K$ be the symplectic resolution of singularities mentioned above.*

- *The pullback*

$$\pi^*: H^{2k}(K, \mathbb{Q}) \rightarrow H^{2k}(\widetilde{K}, \mathbb{Q})$$

is injective for any k .

- *The cohomology groups*

$$H^{2k}(K, \mathbb{Q}) \text{ and } H^{2k+1}(K, \mathbb{Q})$$

carry a pure Hodge structure of weight $2k$.

- *We have the following Betti numbers of K :*

	b_0	b_1	b_2	b_3	b_{10}	b_{11}	b_{12}
K	1	0	7	0	7	0	1

- *We have $b_5(K) \neq 0$.*

- *The Euler characteristic of K is $\chi(K) = 1208$.*

Also in this case, we determine some bounds on the remaining Betti numbers, cfr. Corollary 6.7 and Proposition 6.10. Observe that again we do not exclude that most of the odd-cohomology groups are trivial, and indeed at the moment the only non-trivial one is $H^5(K, \mathbb{Q})$. The reader should not be surprised of the fact that $H^5(K, \mathbb{Q})$ carries a Hodge structure of the “wrong” weight (see Example 6.8).

Remark 1.3. Looijenga, Lunts [15] and Verbitsky [24] studied the cohomology ring of a Hyperkähler manifold X as a representation of a Lie algebra, named after them LLV algebra, isomorphic to $\mathfrak{so}(V, q)$, where

$$(V, q) = \left(H^2(X, \mathbb{Q}) \oplus \mathbb{Q}^2, q_{BB} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

is the extended Mukai lattice. Here q_{BB} is the Beauville-Bogomolov form of X . The decomposition in irreducible representations of $H^*(\widetilde{M}, \mathbb{Q})$ and $H^*(\widetilde{K}, \mathbb{Q})$ is known [9, Theorem 1.2] and one direct summand consists of the cohomology algebra generated by $H^2(\widetilde{M}, \mathbb{Q})$ and $H^2(\widetilde{K}, \mathbb{Q})$ respectively, the so-called Verbitsky component [9, Definition 2.19]. Our results on the Betti numbers of M and K have been obtained by analysing just the intersection of the Verbitsky component of \widetilde{M} and \widetilde{K} with the image of the pullback $\pi^* : H^*(M, \mathbb{Q}) \rightarrow H^*(\widetilde{M}, \mathbb{Q})$ and $\pi^* : H^*(K, \mathbb{Q}) \rightarrow H^*(\widetilde{K}, \mathbb{Q})$ respectively; in future work we plan to investigate the other components too (cfr. Remark 5.12).

Moreover, an LLV algebra for the intersection cohomology of varieties admitting a symplectic resolution of singularities has been recently introduced and computed [7, Theorem 0.5]. It would be interesting to compute their decompositions in irreducible representations of the intersection cohomology of M and K with respect to their LLV algebras.

Corollary 1.4. *The singularities of the varieties M and K are **not** finite quotient singularities.*

Proof. Theorem 1.1 shows that Poincaré duality fails for $H^4(M, \mathbb{Q})$ and $H^{16}(M, \mathbb{Q})$, and Theorem 1.2 shows that the k^{th} -rational cohomology group of K does not always carry a pure Hodge structure of weight k , as $H^5(K, \mathbb{Q})$ carries a non trivial Hodge structure of weight 4. This is enough to conclude that M and K do not have finite quotient singularities, see [4, §1.3]. \square

The varieties M and K are known to be \mathbb{Q} -factorial [20, Theorem 1.1]. Bakker and Lehn asked about the existence of \mathbb{Q} -factorial symplectic varieties not having finite quotient singularities. This is an interesting class of varieties in the framework of the recent results on Torelli theorems for singular symplectic varieties: the surjectivity of the period map has been proved by Bakker and Lehn [2, Theorem 1.1] for primitive symplectic varieties with \mathbb{Q} -factorial singularities and 2nd Betti number greater or equal than 5, and by Menet [16, Theorem 1.1] without assumption on the 2nd Betti

number but with the further assumption that the singularities are of finite quotient type.

Remark 1.5. It is an easy consequence of [12, Remark 6.3] and [13, Proposition 5.15] that the moduli spaces $M_v(X, H)$ such that $v = kw$ with $k \geq 3$ or $w^2 > 2$ are examples of \mathbb{Q} -factorial symplectic varieties not having finite quotient singularities, as they are not analytically \mathbb{Q} -factorial.

Notice that the case of the varieties M and K is different, as they are also analytically \mathbb{Q} -factorial, see Remark 2.1. Furthermore, the varieties M and K give examples of (analytically) \mathbb{Q} -factorial varieties not having quotient singularities but having a symplectic resolution, differently from the above mentioned $M_v(X, H)$, [12, Theorem 6.2].

Notations. We will always work over the field of complex numbers \mathbb{C} . Given a compact complex variety X we will denote by $H^k(X)$ its k^{th} -cohomology group with rational coefficients.

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2. THE RESOLUTION OF THE SINGULARITY OF O'GRADY

Consider the Mukai vector $v = (2, 0, -2)$ and the moduli space $M_v(X, H)$ as in the introduction, with X a K3 or an abelian surface. The singular points of $M_v(X, H)$ are the S -equivalence classes of strictly semistable sheaves; if we write $v = 2v_0$, the singular locus of $M_v(X, H)$ is isomorphic to the symmetric product $\text{Sym}^2 M_{v_0}(X, H)$, with singular locus isomorphic to $M_{v_0}(X, H)$.

Let Y be one of O'Grady's singular moduli spaces M and K as in the introduction, introduced by O'Grady in [18, 19]. We call $\Sigma := Y^{\text{sing}}$ and $\Omega := \Sigma^{\text{sing}}$. If $X = S$ is a K3 surface and $Y = M$ then $\Omega \cong M_{v_0}(S, H)$ is deformation equivalent to the Hilbert scheme of two points on a K3 surface, because the vector v_0 is primitive in the Mukai lattice $H^{2*}(S, H)$ of S . If $X = A$ is an abelian surface, then $M_{v_0}(A, H) \cong A \times A^\vee$ and in $Y = K$ we have that $\Sigma = K \cap \text{Sym}^2 M_{v_0}(A, H)$ is isomorphic to $(A \times A^\vee) / \pm 1$, with singular locus Ω consisting of 256 points. For any detail we refer to [21, §1] and [17, §1].

Finally, we call $\pi : \tilde{Y} := \text{Bl}_\Sigma Y \rightarrow Y$ the blow-up giving the symplectic resolution as discussed in the introduction.

Remark 2.1. The varieties M and K are analytically \mathbb{Q} -factorial. Indeed, they have A_1 -singularities along $\Sigma \setminus \Omega$ (cfr. the local description below) and they have locally factorial singularities along Ω [12, Remark 6.3,(2)].

2.2. Local structure. The local structure of the singularities of O'Grady's moduli spaces was described by M. Lehn and Sorger [14]. Mongardi, Rapagnetta and Saccà [17] gave a description of the local structure in terms of blow-ups along smooth subvarieties. We briefly recall these descriptions here and refer to the original works for details.

Let (V, ω) be a complex symplectic vector space of dimension 4 and denote by $\mathfrak{sp}(V)$ the Lie algebra of the symplectic group of (V, ω) . We define

$$(2.1) \quad Z := \{A \in \mathfrak{sp}(V) \mid A^2 = 0\}.$$

These are exactly the endomorphisms of $\mathfrak{sp}(V)$ of rank ≤ 2 and the singular locus of Z is given by $\Sigma_Z = \{A \in Z \mid \text{rk}(A) \leq 1\}$; the singular locus of Σ_Z is $\Omega_Z = \{0\}$. Furthermore, Z has A_1 -singularities along Σ_Z .

Theorem 2.3 ([14, Théorème 4.5]). *There are isomorphisms of analytic germs:*

- (1) $(K, p) \cong (Z, 0)$, for any point $p \in \Omega \subset K$.
- (2) $(M, p) \cong (Z \times \mathbb{C}^4, 0)$, for any point $p \in \Omega \subset M$.

We denote by G the Lagrangian Grassmanian of V . We put

$$(2.2) \quad \tilde{Z} := \{(A, U) \in Z \times G \mid U \subset \ker A\} \subseteq Z \times G$$

and call $\pi_Z : \tilde{Z} \rightarrow Z$ the restriction of the projection on the first factor. The restriction of the second projection $\tilde{Z} \rightarrow G$ makes \tilde{Z} the cotangent bundle of G , hence \tilde{Z} is smooth and $\pi_Z : \tilde{Z} \rightarrow Z$ is a symplectic resolution of Z . Lehn and Sorger show the following.

Theorem 2.4 ([14, Théorème 2.1]). *The morphism $\pi_Z : \tilde{Z} \rightarrow Z$ defined above coincides with the blow up of Z in Σ_Z equipped with the reduced structure.* \square

We call $\tilde{\Sigma}_Z$ the exceptional divisor of π_Z , i.e.

$$\tilde{\Sigma}_Z = \{(A, U) \in \Sigma_Z \times G \mid U \subset \ker A\}.$$

The restriction morphism $\pi_Z|_{\tilde{\Sigma}_Z} : \tilde{\Sigma}_Z \rightarrow \Sigma_Z$ is a \mathbb{P}^1 -fiber bundle outside of Ω_Z , whereas the fiber $\tilde{\Omega}_Z := \pi_Z^{-1}(0) = G$ is the Lagrangian Grassmanian.

As observed in [17, Remark 2.1], the variety \tilde{Z} is isomorphic to the total space $\text{Sym}_G^2 \mathcal{U}$, where \mathcal{U} is the rank 2 tautological bundle over G ; the isomorphism is realized as follows. Given $(A, U) \in \tilde{\Sigma}_Z$, the endomorphism A factorizes as $V \rightarrow V/U \xrightarrow{\phi_A} U \hookrightarrow V$, and the symplectic form ω induces an isomorphism $V/U \cong U^\vee$ because U is a Lagrangian subspace. Hence $\phi_A \in \text{Hom}(U^\vee, U) \cong U \otimes U \cong (U^\vee \otimes U^\vee)^\vee$, where the associated bilinear form $(f, g) \mapsto f(\phi_A(g))$ on $U^\vee \otimes U^\vee$ is symmetric because $A \in \mathfrak{sp}(V)$ satisfies $\omega(Av, w) = \omega(Aw, v)$ for any $v, w \in V$; hence $\phi_A \in \text{Sym}^2 U$. Under this identification the variety $\tilde{\Sigma}$ corresponds to the locus parametrizing singular

symmetric bilinear forms on the fibers of \mathcal{U}^\vee , which is a fibration in cones over a smooth conic over G , having singularities only along the zero section $\tilde{\Omega}_Z$.

Following [17, Proposition 2.4] we consider the varieties:

$$\begin{aligned}\bar{Z} &:= \text{Bl}_{\Omega_Z} Z = \{([B], A) \mid A \in [B]\} \subseteq \mathbb{P}(Z) \times Z \\ \hat{Z} &:= \text{Bl}_{\tilde{\Omega}_Z} \tilde{Z} = \{([B], A, U) \mid A(U) = 0, A \in [B]\} \subseteq \mathbb{P}(Z) \times Z \times G\end{aligned}$$

with blow-up morphisms $\rho_Z : \bar{Z} \rightarrow Z$ and $\gamma_Z : \hat{Z} \rightarrow \tilde{Z}$. If $\bar{\Sigma}_Z$ is the strict transform of Σ_Z in \bar{Z} via ρ_Z , i.e. $\bar{\Sigma}_Z \cong \text{Bl}_{\Omega_Z} \Sigma_Z$, one has $\hat{Z} \cong \text{Bl}_{\bar{\Sigma}_Z} \bar{Z}$ and the following diagram is commutative:

$$\begin{array}{ccc} & \hat{Z} & \\ \gamma_Z \swarrow & & \searrow \phi_Z \\ \tilde{Z} & & \bar{Z} \\ \pi_Z \searrow & & \swarrow \rho_Z \\ & Z & \end{array}$$

Here $\phi_Z : \hat{Z} \rightarrow \bar{Z}$ is the blow-up morphism. Furthermore, $\bar{\Sigma}_Z$ is smooth, \bar{Z} has A_1 singularity along $\bar{\Sigma}_Z$ and \hat{Z} is smooth. Observe that the morphisms ϕ_Z , γ_Z and ρ_Z are blow-ups along smooth subvarieties, and the diagram above relates the resolution $\pi_Z : \tilde{Z} \rightarrow Z$ with them. This is exactly in the spirit of the original works [18] and [19] by O'Grady.

We give names to the transformations of Ω_Z and Σ_Z inside the new varieties \bar{Z} and \hat{Z} : We call $\bar{\Omega}_{OG,Z}$ the exceptional divisor of $\bar{Z} \xrightarrow{\rho_Z} Z$ and we set $\bar{\Omega}_Z := \bar{\Omega}_{OG,Z} \cap \bar{\Sigma}_Z$; note that $\bar{\Omega}_Z = \{([B], 0) \mid [B] \in \mathbb{P}(\Sigma_Z)\} \cong \mathbb{P}(V)$. We denote by $\hat{\Sigma}_Z$ the exceptional divisor of $\hat{Z} \xrightarrow{\phi_Z} \bar{Z}$, by $\hat{\Omega}_{OG,Z}$ the exceptional divisor of $\hat{Z} \xrightarrow{\gamma_Z} \tilde{Z}$ and we set $\hat{\Omega}_Z := \hat{\Omega}_{OG,Z} \cap \hat{\Sigma}_Z \subseteq \hat{\Omega}_{OG,Z}$.

Proposition 2.5. [17, Corollary 2.5]

- (1) $\hat{\Sigma}_Z$ is a \mathbb{P}^1 -bundle over $\bar{\Sigma}_Z$ and $\hat{\Sigma}_Z \cong \text{Bl}_{\tilde{\Omega}_Z} \tilde{\Sigma}_Z$.
- (2) $\hat{\Omega}_{OG,Z}$ is a \mathbb{P}^2 -bundle over $\tilde{\Omega}_Z$ isomorphic to $\mathbb{P}(\text{Sym}_G^2 \mathcal{U})$.

The second part of the proposition above is obtained via the isomorphism $\tilde{Z} \cong \text{Sym}^2 \mathcal{U}$ described before, which realizes $\tilde{\Omega}_Z$ as the zero section of $\text{Sym}^2 \mathcal{U}$; as consequence $\hat{Z} = \text{Bl}_{\tilde{\Omega}_Z} \tilde{Z}$ is isomorphic to the total space $T \subseteq \mathbb{P}(\text{Sym}_G^2 \mathcal{U}) \times \text{Sym}_G^2 \mathcal{U}$ and it has exceptional divisor $\hat{\Omega}_{OG,Z} \cong \mathbb{P}(\text{Sym}_G^2 \mathcal{U})$.

Corollary 2.6. $\hat{\Omega}_Z$ is a \mathbb{P}^1 -bundle over $\tilde{\Omega}_Z$ isomorphic to $\mathbb{P}_G(\mathcal{U})$, and the natural inclusion $\hat{\Omega}_Z \subseteq \hat{\Omega}_{OG,Z}$ corresponds to the inclusion $\mathbb{P}_G(\mathcal{U}) \subseteq \mathbb{P}(\text{Sym}_G^2 \mathcal{U})$ induced by $U \rightarrow \text{Sym}^2 U$, $u \mapsto u \otimes u$.

Proof. From Proposition 2.5 we have that $\hat{\Omega}_Z$ is the exceptional divisor of $\hat{\Sigma}_Z \cong \text{Bl}_{\tilde{\Omega}_Z} \tilde{\Sigma}_Z$. An element in Σ_Z is of the form $A_v = \omega(-, v)v$ for some $v \in U$; the isomorphism $\tilde{Z} \cong \text{Sym}_G^2 \mathcal{U}$ sends the element $(A_v, U) \in \tilde{\Sigma}_Z$ to the symmetric bilinear form $\{(f, g) \mapsto f(v)g(v)\}$ on U^\vee , which lies in the image of $U \xrightarrow{\alpha} U \otimes U \cong (U^\vee \otimes U^\vee)^\vee$,

where α is the map $u \mapsto u \otimes u$. Notice that two such symmetric bilinear form $\{(f, g) \mapsto f(v)g(v)\}$ and $\{(f, g) \mapsto f(w)g(w)\}$ are equal exactly when $w = \pm v$, which is the condition to have $A_v = A_w$. We conclude $\tilde{\Sigma}_Z \cong \alpha_G(\mathcal{U}) \cong \mathcal{U}/\pm 1 \subseteq \text{Sym}_G^2 \mathcal{U} \cong \tilde{Z}$, where α_G is the relative version of the α just described on the fibers and ± 1 is the fiberwise action; again, $\tilde{\Omega}_Z$ corresponds to the zero section of \mathcal{U} . Hence $\hat{\Sigma}_Z \cong \text{Bl}_{\tilde{\Omega}_Z} \tilde{\Sigma}_Z$ is isomorphic to the total space $T \subseteq \mathbb{P}_G(\mathcal{U}) \times \mathcal{U}/\pm 1$ and it has exceptional divisor $\hat{\Omega}_Z \cong \mathbb{P}_G(\mathcal{U})$, with inclusion $\hat{\Omega}_Z = \hat{\Omega}_{OG,Z} \cap \hat{\Sigma}_Z \subseteq \hat{\Omega}_{OG,Z}$ as in the statement. \square

Proposition 2.7. $\hat{\Omega}_Z$ is a \mathbb{P}^1 -bundle over $\bar{\Omega}_Z$ isomorphic to $\mathbb{P}_{\bar{\Omega}_Z}(\mathcal{L}^\perp/\mathcal{L})$, where \mathcal{L} is the tautological line bundle of $\bar{\Omega}_Z \cong \mathbb{P}(V)$.

Proof. We define an isomorphism $\mathbb{P}_G(\mathcal{U}) \xrightarrow{\sim} \mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^\perp/\mathcal{L})$, and the claim will follow from Corollary 2.6. Given a Lagrangian subspace U of V and a line L in U , then U is contained in L^\perp and defines a line \bar{U} in the quotient L/L^\perp ; the association $(L, U) \mapsto (L, \bar{U})$ defines the desired morphism. On the other hand, given a line L in V and a line W in L^\perp/L , then the preimage U of W in L along the quotient map L^\perp/L is a Lagrangian subspace of V which contains L . Thus, the association $(L, W) \mapsto (L, U)$ defines the inverse morphism. \square

2.8. Global structure. Here we state the global version of the results in the previous section. In analogy with the constructions and notations used in the local setting, we consider:

- $\bar{Y} := \text{Bl}_{\bar{\Omega}} Y \xrightarrow{\rho} Y$ with exceptional divisor $\bar{\Omega}_{OG}$; we call $\bar{\Sigma}$ the strict transform of Σ and $\bar{\Omega} := \bar{\Omega}_{OG} \cap \bar{\Sigma}$.
- $\hat{Y} := \text{Bl}_{\hat{\Sigma}} \bar{Y} \xrightarrow{\phi} \bar{Y}$ with exceptional divisor $\hat{\Sigma}$.
- $\text{Bl}_{\hat{\Omega}} \tilde{Y} \xrightarrow{\gamma} \tilde{Y}$ with exceptional divisor $\hat{\Omega}_{OG}$; we call $\hat{\Omega} := \hat{\Omega}_{OG} \cap \hat{\Sigma}$.

The varieties here introduced have been firstly considered by O'Grady [18, 19]. The following result is [17, Proposition 2.6] when $Y = K$, and exactly the same proof gives the result when $Y = M$.

Proposition 2.9. *The varieties \hat{Y} and $\text{Bl}_{\hat{\Omega}} \tilde{Y}$ are smooth and isomorphic over Y , hence the following diagram is commutative:*

$$\begin{array}{ccc}
 & \hat{Y} & \\
 \gamma \swarrow & & \searrow \phi \\
 \tilde{Y} & & \bar{Y} \\
 \pi \searrow & & \swarrow \rho \\
 & Y &
 \end{array}$$

We pass to the global structure of the Σ -varieties. For similar statements we refer to [18, 19] and to [17, Remark 2.7, Corollary 2.8(1)] for the case $Y = K$.

Proposition 2.10. *Let Y be the variety M or K .*

- (1) $q := \rho|_{\bar{\Sigma}} : \bar{\Sigma} \rightarrow \Sigma$ is the blow-up of Σ in Ω , with exceptional divisor $\bar{\Omega}$.

- (2) $f := \phi|_{\widehat{\Sigma}} : \widehat{\Sigma} \rightarrow \overline{\Sigma}$ is a \mathbb{P}^1 -fiber bundle.
- (3) $g := \gamma|_{\widehat{\Sigma}} : \widehat{\Sigma} \rightarrow \widetilde{\Sigma}$ is the blow-up of $\widetilde{\Sigma}$ in $\widetilde{\Omega}$, with exceptional divisor $\widehat{\Omega}$.

In particular, the varieties $\overline{\Sigma}$ and $\widehat{\Sigma}$ are smooth.

- Proof.* (1) The statement follows from the fact that Σ contains Ω as closed subscheme. $\overline{\Sigma}$ is smooth because $\Sigma \setminus \Omega$ is smooth and Y has A_1 singularities along it, as it follows from the local description of the varieties.
- (2) The statement follows from the local one in Proposition 2.5(1).
 - (3) The statement follows again from Proposition 2.5(1), as the blow-up is a local construction.

□

We conclude with the global structure of the Ω -varieties.

Proposition 2.11. *Consider the case $Y = M$.*

- (1) $q_{\Omega} := q|_{\overline{\Omega}} : \overline{\Omega} \rightarrow \Omega$ is a \mathbb{P}^3 -bundle. More precisely, $\overline{\Omega} \cong \mathbb{P}_{\Omega}(T\Omega)$ over Ω .
- (2) $f_{\Omega} := f|_{\widehat{\Omega}} : \widehat{\Omega} \rightarrow \overline{\Omega}$ is a \mathbb{P}^1 -bundle. More precisely, let $\mathcal{L} \subset q_{\Omega}^*T\Omega$ be the tautological subbundle, then $\mathbb{P}_{\Omega}(\mathcal{L}^{\perp}/\mathcal{L}) \simeq \widehat{\Omega}$ over $\overline{\Omega}$.
- (3) $p_{\Omega} := \pi|_{\widetilde{\Omega}} : \widetilde{\Omega} \rightarrow \Omega$ is a G -bundle, where G is the Lagrangian Grassmannian of 2 dimensional Lagrangian spaces in a symplectic vector space. More precisely, $\widetilde{\Omega} \cong \mathcal{L}\mathcal{G}_{\Omega}(T\Omega)$ over Ω , where $\mathcal{L}\mathcal{G}_{\Omega}(T\Omega)$ is the relative Lagrangian Grassmannian on $T\Omega$.
- (4) $g_{\Omega} := g|_{\widehat{\Omega}} : \widehat{\Omega} \rightarrow \widetilde{\Omega}$ is a \mathbb{P}^1 -bundle. More precisely, $\widehat{\Omega} \cong \mathbb{P}(\mathcal{U})$ over $\widetilde{\Omega}$, where \mathcal{U} is the tautological bundle of $\mathcal{L}\mathcal{G}_{\Omega}(T\Omega)$.

In particular, the four varieties Ω , $\overline{\Omega}$, $\widetilde{\Omega}$ and $\widehat{\Omega}$ are smooth.

- Proof.* (1) Since $\Sigma \cong \text{Sym}^2\Omega$, the blow-up (cfr. Proposition 2.10.(1)) $\overline{\Sigma} \xrightarrow{q} \Sigma$ can be identified with the Hilbert-Chow morphism $\Omega^{[2]} \rightarrow \text{Sym}^2\Omega$. The exceptional divisor $\overline{\Omega}$ is then isomorphic to $\mathbb{P}(T\Omega)$.
- (2) This follows essentially from the construction of O'Grady and the careful analysis of the various blow-ups. Since the details are technical, we discuss this point in the appendix for sake of the exposition.
 - (3) This follows by the original construction of \widetilde{M} in O'Grady's work, see [18, §2.2].
 - (4) By [18, Proposition 2.0.1] the variety $\widehat{\Omega}_{OG}$ is isomorphic to $\mathbb{P}_{\widetilde{\Omega}}(\text{Sym}^2\mathcal{U})$ over $\widetilde{\Omega} \cong \mathcal{L}\mathcal{G}_{\Omega}(T\Omega)$, hence from the local description in Corollary 2.6 we deduce that $\widehat{\Omega}$ is identified with the rank 1 tensors in $\mathbb{P}_{\widetilde{\Omega}}(\text{Sym}^2\mathcal{U})$, thus isomorphic to $\mathbb{P}_{\widetilde{\Omega}}(\mathcal{U})$.

As Ω is smooth, all the Ω -varieties are smooth as well.

□

Now, let A be an abelian surface. The dimension 10 variety $M_v(A, H)$ has an up to scalar unique symplectic form σ on its regular part, which by [11, Theorem 2.3] induces a symplectic form α on $M_{v_0}(A, H)$. Let V be the restriction $TM_{v_0}(A, H)|_{\Omega}$ of its tangent bundle, which we consider as a symplectic vector bundle equipped with the restriction of α .

3. SEMISIMPLICIAL RESOLUTIONS

The mixed Hodge structure of a complete algebraic variety can be computed using semisimplicial resolutions [6]. Following the algorithm in the proof of [22, Theorem 5.26], we obtain a semisimplicial resolution from a cubical hyperresolution as we will now explain.

Assume the notations are as in Notation 2.13 and consider the diagram

$$(3.1) \quad \begin{array}{ccccc} \tilde{Y} & \xleftarrow{\tilde{j}} & \tilde{\Sigma} & \xleftarrow{\tilde{i}} & \tilde{\Omega} \\ \pi \downarrow & & p \downarrow & & p_{\Omega} \downarrow \\ Y & \xleftarrow{j} & \Sigma & \xleftarrow{i} & \Omega \end{array}$$

As observed in Section 2 the morphism $p : \tilde{\Sigma} \rightarrow \Sigma$ is generically a smooth \mathbb{P}^1 -bundle over the complement of Ω and $p_{\Omega} : \tilde{\Omega} \rightarrow \Omega$ is a LG_2 -fiber bundle. To obtain a semisimplicial resolution, we consider the resolution $q : \bar{\Sigma} = \text{Bl}_{\Omega}\Sigma \rightarrow \Sigma$ (cfr. Proposition 2.10) and the induced diagram

$$(3.2) \quad \begin{array}{ccccc} & & \hat{\Sigma} & \xleftarrow{\hat{i}} & \hat{\Omega} \\ & g \swarrow & \downarrow f & & \searrow g_{\Omega} \\ \tilde{\Sigma} & \xleftarrow{\tilde{i}} & \tilde{\Omega} & & \downarrow f_{\Omega} \\ \downarrow p & & \downarrow p_{\Omega} & & \downarrow \\ \Sigma & \xleftarrow{i} & \bar{\Sigma} & \xleftarrow{\bar{i}} & \bar{\Omega} \\ & q \swarrow & \downarrow q_{\Omega} & & \searrow \\ & & \Sigma & \xleftarrow{i} & \Omega \end{array}$$

Let $Y_{\{\bullet\}}$ be the 2-cubical variety

$$(3.3) \quad \begin{array}{ccccc} & & \hat{\Sigma} & \xleftarrow{\hat{i}} & \hat{\Omega} \\ & \tilde{j} \circ g \swarrow & \downarrow f & & \searrow g_{\Omega} \\ \tilde{Y} & \xleftarrow{\tilde{i}} & \tilde{\Omega} & & \downarrow f_{\Omega} \\ \downarrow \pi & & \downarrow p_{\Omega} & & \downarrow \\ Y & \xleftarrow{i} & \bar{\Sigma} & \xleftarrow{\bar{i}} & \bar{\Omega} \\ & q \swarrow & \downarrow q_{\Omega} & & \searrow \\ & & \Sigma & \xleftarrow{i} & \Omega \end{array}$$

and let $\varepsilon_{\bullet} : Y_{\bullet} \rightarrow Y$ be the induced augmented semisimplicial variety

$$(3.4) \quad \begin{array}{ccccc} Y_0 & \xleftarrow{\varepsilon_0} & Y_1 & \xleftarrow{\varepsilon_1} & Y_2 \\ \varepsilon_0 \downarrow & & \varepsilon_1 \swarrow & & \searrow \varepsilon_2 \\ Y & & Y & & Y \end{array}$$

where

$$Y_0 = \tilde{Y} \amalg \tilde{\Sigma} \amalg \Omega, \quad Y_1 = \hat{\Sigma} \amalg \tilde{\Omega} \amalg \bar{\Omega}, \quad Y_2 = \hat{\Omega}.$$

Proposition 3.1. *The augmented semisimplicial variety is a semisimplicial hyperresolution of Y .*

Proof. The very construction of $\varepsilon_\bullet : Y_\bullet \rightarrow Y$ here above is done following the algorithm in the proof of [22, Theorem 5.26]. In our case going through the steps of the algorithm, we first consider the resolution \tilde{Y} of $Y_\emptyset^{(1)} := Y$ together with its exceptional locus and discriminant, which form the diagram

$$\begin{array}{ccc} \tilde{Y} & \longleftarrow & \tilde{\Sigma} \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \Sigma. \end{array}$$

As the the 2-cubical variety $\tilde{\Sigma} \rightarrow \Sigma$ is singular, we take the resolution of singularities (of 2-cubical varieties)

$$\begin{array}{ccc} \tilde{\Sigma} & \longleftarrow & \hat{\Sigma} \\ \downarrow & & \downarrow \\ \Sigma & \longleftarrow & \bar{\Sigma}. \end{array}$$

and we complete the diagram with the discriminant and exceptional locus so to get the diagram

$$(3.5) \quad \begin{array}{ccccccc} & & Y_{\{0,1\}} := \hat{\Sigma} & \xleftarrow{\hat{i}} & Y_{\{0,1,2\}} := \hat{\Omega} & & \\ & & \downarrow f & & \swarrow g_\Omega & & \\ Y_{\{0\}} := \tilde{Y} & \longleftarrow & \tilde{\Sigma} & \xleftarrow{\tilde{i}} & Y_{\{0,2\}} := \tilde{\Omega} & & \\ \downarrow \pi & & \downarrow \pi_\Sigma & & \downarrow p_\Omega & & \\ & & Y_{\{1\}} := \bar{\Sigma} & \xleftarrow{\bar{i}} & Y_{\{1,2\}} := \bar{\Omega} & & \\ & & \downarrow & & \swarrow q_\Omega & & \\ Y_\emptyset := Y & \longleftarrow & \Sigma & \xleftarrow{i} & Y_{\{2\}} := \Omega & & \end{array}$$

Suppressing the singular 2-variety $\tilde{\Sigma} \rightarrow \Sigma$ we finally get to diagram (3.3), which is a cubical hyperresolution as the smoothness of the Y_i follows from Proposition 2.10, Proposition 2.12 and Proposition 2.11. From the cubical hyperresolution one constructs the semisimplicial resolution of Y by declaring

$$Y_0 = Y_{\{0\}} \amalg Y_{\{1\}} \amalg Y_{\{2\}}, \quad Y_1 = Y_{\{0,1\}} \amalg Y_{\{0,2\}} \amalg Y_{\{1,2\}}, \quad Y_2 = Y_{\{0,1,2\}}$$

together with morphisms as in diagram (3.4) using the arrows in the cube. In particular one builds two morphisms $s_0, s_1 : Y_1 \rightarrow Y_0$: the first by taking only the arrows in the

cube corresponding to inclusions of singletons in first position, i.e.

$$\{0\} \subset \{0, 1\}, \quad \{0\} \subset \{0, 2\}, \quad \{1\} \subset \{1, 2\}$$

and the second using the remaining inclusions

$$\{1\} \subset \{0, 1\}, \quad \{2\} \subset \{0, 2\}, \quad \{2\} \subset \{1, 2\}.$$

□

From the semisimplicial hyperresolution we get the spectral sequence [10, Proposition 3.3]

$$(3.6) \quad E_1^{p,q} = H^q(Y_p, \mathbb{Q}) \Rightarrow H^{p+q}(Y, \mathbb{Q})$$

which degenerates at the second page and converges to the cohomology of Y filtered by the weights. The differentials of the first page are given by an alternating sum of pullbacks along the maps appearing in the resolution taken with a sign. Spelling out the details in our case, we get that the differential $d_0 := s_0^* - s_1^*: E_1^{0,q} \rightarrow E_1^{1,q}$ is given by the sum of pullbacks

$$(3.7) \quad \begin{array}{ccc} H^q(\tilde{Y}) & \xrightarrow{(\tilde{j} \circ g)^*} & H^q(\widehat{\Sigma}) \\ & \begin{array}{c} \searrow \tilde{i}^* \\ \nearrow f^* \end{array} & \\ H^q(\overline{\Sigma}) & & H^q(\tilde{\Omega}) \\ & \begin{array}{c} \searrow \bar{i}^* \\ \nearrow p_\Omega^* \end{array} & \\ H^q(\Omega) & \xrightarrow{q_\Omega^*} & H^q(\overline{\Omega}). \end{array}$$

where the red ones are taken with a minus sign. Thus, one sees at once that

Lemma 3.2.

$$\begin{aligned} E_2^{0,q} = \{ & (a, b, c) \in H^q(\tilde{Y}) \oplus H^q(\overline{\Sigma}) \oplus H^q(\Omega) \mid (\tilde{j} \circ g)^* a - f^* b = 0 \in H^q(\widehat{\Sigma}), \\ & \tilde{i}^* a - p_\Omega^* c = 0 \in H^q(\tilde{\Omega}), \\ & \bar{i}^* b - q_\Omega^* c = 0 \in H^q(\overline{\Omega}) \}. \end{aligned}$$

On the other hand, the differential $d_1 : E_1^{1,q} \rightarrow E_1^{2,q}$ is the difference of blue and red

$$(3.8) \quad \begin{array}{ccc} & H^q(\widehat{\Sigma}) & \\ & \searrow \hat{i}^* & \\ H^q(\tilde{\Omega}) & \xrightarrow{g_\Omega^*} & H^q(\widehat{\Omega}). \\ & \nearrow f_\Omega^* & \\ & H^q(\overline{\Omega}) & \end{array}$$

hence one has the following

Lemma 3.3. $E_2^{2,q}$ is the cokernel of

$$d_1: H^q(\widehat{\Sigma}) \oplus H^q(\overline{\Omega}) \oplus H^q(\overline{\Omega}) \longrightarrow H^q(\widehat{\Omega})$$

$$(a, b, c) \longmapsto \widehat{i}^*a - g_{\Omega}^*b - f_{\Omega}^*c.$$

The following sections are devoted to the computation of the page $E_2^{p,q}$ when $Y = M$ and K .

4. THE COHOMOLOGY OF THE Σ -VARIETIES AND Ω -VARIETIES

Assume the notations as in Notation 2.13. We start the computation of the spectral sequence (3.6) computing the objects in the page $E_1^{p,q}$, i.e. the cohomology of the Ω - and Σ -varieties.

4.1. The cohomology of the Ω -varieties. We start describing the cohomology of the Ω -varieties in the local case presented in Section 2.2: Let (V, ω) be a symplectic vector space of dimension 4, $G = LG(V)$ be the Lagrangian Grassmannina of it with tautological bundle \mathcal{U} and $\mathcal{L} \subseteq V \otimes \mathcal{O}_{\mathbb{P}(V)}$ be the tautological line subbundle. We have described the following local structure of the Ω -varieties:

$$\begin{array}{ccc}
 & \widehat{\Omega}_Z & \\
 \swarrow \gamma_{\Omega} & & \searrow \phi_{\Omega} \\
 \widetilde{\Omega}_Z & & \overline{\Omega}_Z \\
 \searrow \pi_{\Omega} & & \swarrow \rho_{\Omega} \\
 & \Omega_Z &
 \end{array}
 \cong
 \begin{array}{ccc}
 & \mathbb{P}_G(\mathcal{U}) \cong \mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^{\perp}/\mathcal{L}) & \\
 \swarrow \gamma_{\Omega} & & \searrow \phi_{\Omega} \\
 G & & \mathbb{P}(V) \\
 \searrow \pi_{\Omega} & & \swarrow \rho_{\Omega} \\
 & \{0\} &
 \end{array}$$

where the maps are the restriction of the respective ones to the Ω_Z -varieties. Consider the tautological short exact sequence on G

$$0 \rightarrow \mathcal{U} \rightarrow V \otimes \mathcal{O}_G \rightarrow \mathcal{Q} \rightarrow 0.$$

Since $U \in G$ is Lagrangian we have $V/U \cong U^{\vee}$, hence $\mathcal{Q} \cong \mathcal{U}^{\vee}$ and

$$c_1(\mathcal{U})^2 - 2c_2(\mathcal{U}) = c_2(\mathcal{U})^2 = 0 \in H^*(G).$$

As G is a 3-dimensional quadric we conclude that

$$H^*(G) \simeq \frac{\mathbb{Q}[u_1, u_2]}{(u_1^2 - 2u_2, u_2^2)},$$

where the isomorphism identifies u_1 and u_2 with the first and the second Chern classes of \mathcal{U} , which have degree 1 and 2.

The pullback along γ_{Ω} defines an injective morphism of rings $H^*(G) \rightarrow H^*(\mathbb{P}(\mathcal{U}))$ and we have

$$H^*(\mathbb{P}_G(\mathcal{U})) \simeq \frac{H^*(G)[\xi]}{(\xi^2 - c_1(\mathcal{U})\xi + c_2(\mathcal{U}))} \simeq \frac{\mathbb{Q}[u_1, u_2, \xi]}{(u_1^2 - 2u_2, u_2^2, \xi^2 - u_1\xi + u_2)},$$

where the first isomorphism takes the first Chern class of the tautological subbundle of $\gamma_{\Omega}^*\mathcal{U}$ to ξ .

The cohomology ring of the projective space $\mathbb{P}(V)$ is generated by the first Chern class of \mathcal{L} and we have

$$H^*(\mathbb{P}(V)) \simeq \frac{\mathbb{Q}[\zeta]}{(\zeta^4)}, \quad c_1(\mathcal{L}) \mapsto \zeta.$$

Next, we look at $\phi_\Omega: \mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^\perp/\mathcal{L}) \rightarrow \mathbb{P}(V)$ and we denote by h the first Chern class of the tautological line subbundle of $\phi_\Omega^*(\mathcal{L}^\perp/\mathcal{L})$; the projective bundle formula gives

$$H^*(\mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^\perp/\mathcal{L})) \simeq \frac{\phi_\Omega^* H^*(\mathbb{P}(V))[h]}{(h^2 - hc_1(\mathcal{L}^\perp/\mathcal{L}) + c_2(\mathcal{L}^\perp/\mathcal{L}))}.$$

Using the two short exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{L}^\perp \rightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{L}^\vee \rightarrow 0 \\ 0 \rightarrow \mathcal{L} \rightarrow \mathcal{L}^\perp \rightarrow \mathcal{L}^\perp/\mathcal{L} \rightarrow 0 \end{aligned}$$

one computes

$$(4.1) \quad c(\mathcal{L}^\perp/\mathcal{L}) = c(\mathcal{L}^\perp)/c(\mathcal{L}) = c(V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)})/c(\mathcal{L})c(\mathcal{L}^\vee) = 1 + \zeta^2$$

so that

$$H^*(\mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^\perp/\mathcal{L})) = \frac{\mathbb{Q}[h, \zeta]}{(\zeta^4, h^2 + \zeta^2)}.$$

Finally, we relate the two descriptions above of the cohomology ring of $\mathbb{P}_G(\mathcal{U}) \cong \mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^\perp/\mathcal{L})$ in the following

Proposition 4.2. *Consider the natural isomorphism $\mathbb{P}_G(\mathcal{U}) \cong \mathbb{P}_{\mathbb{P}(V)}(\mathcal{L}^\perp/\mathcal{L})$ defined in Proposition 2.7. The isomorphism induced on their cohomology rings is given by:*

$$\begin{aligned} \xi &\mapsto \zeta \\ u_2 &\mapsto \zeta h \\ u_1 &\mapsto \zeta + h. \end{aligned}$$

Proof. Clearly we have $\xi \mapsto \zeta$. Moreover, consider the tautological short exact sequence over $\mathbb{P}_G(\mathcal{U})$:

$$0 \rightarrow \mathcal{L} \rightarrow \gamma_\Omega^* \mathcal{U} \rightarrow \mathcal{F} \rightarrow 0.$$

As the composition $\gamma_\Omega^* \mathcal{U} \subset \mathcal{L}^\perp \rightarrow \mathcal{L}^\perp/\mathcal{L}$ vanishes on \mathcal{L} , it factors through \mathcal{F} and shows \mathcal{F} as the tautological subbundle of $\mathcal{L}^\perp/\mathcal{L}$. Thus, from the above short exact sequence it follows that $u_1 \mapsto h + \zeta$ and $u_2 \mapsto h\zeta$. \square

This suffices to determine the cohomology of the Ω -varieties in the 6-dimensional variety of O'Grady.

Proposition 4.3. *Consider the case $Y = K$. The cohomology of the Ω -varieties is described as follows.*

$$\begin{aligned} (1) \quad H^*(\tilde{\Omega}) &= \left(\frac{\mathbb{Q}[u_1]}{(u_1^4)} \right)^{\oplus 256}. \\ (2) \quad H^*(\bar{\Omega}) &= \left(\frac{\mathbb{Q}[\zeta]}{(\zeta^4)} \right)^{\oplus 256}. \end{aligned}$$

(3) The cohomology of $\widehat{\Omega}$ is an algebra on the cohomology of $\overline{\Omega}$ and $\widetilde{\Omega}$ via pullback:

$$\left(\frac{\mathbb{Q}[u_1, \xi]}{(u_1^4, \xi^2 - \xi u_1 + u_1^2/2)} \right)^{\oplus 256} = H^*(\widehat{\Omega}) = \left(\frac{\mathbb{Q}[\zeta, h]}{(\zeta^4, h^2 + \zeta^2)} \right)^{\oplus 256}.$$

Along this isomorphism we have the identifications $u_1 \mapsto h + \zeta$ and $\xi \mapsto \zeta$.

Proof. Using Proposition 2.12 the statements follow immediately from the analysis in the local case above. \square

For the 10-dimensional variety of O'Grady we need some extra work.

Proposition 4.4. *Consider the case $Y = M$. The cohomology of the Ω -varieties is described as follows.*

$$\begin{aligned} (1) \quad H^*(\widetilde{\Omega}) &= \frac{H^*(\Omega)[u_1, u_2]}{(-u_1^2 + 2u_2 - c_2(T\Omega), c_4(T\Omega) - u_2^2)}. \\ (2) \quad H^*(\overline{\Omega}) &= \frac{H^*(\Omega)[\zeta]}{(\zeta^4 + \zeta^2 c_2(T\Omega) + c_4(T\Omega))}. \\ (3) \quad \frac{H^*(\Omega)[u_1, u_2, \xi]}{(-u_1^2 + 2u_2 - c_2(T\Omega), u_2^2 - c_4(T\Omega), \xi^2 - \xi u_1 + u_2)} &= H^*(\widehat{\Omega}) = \frac{H^*(\Omega)[h, \zeta]}{(\zeta^4 + \zeta^2 c_2(T\Omega) + c_4(T\Omega), \zeta^2 + h^2 + c_2(T\Omega))}. \end{aligned}$$

Along this isomorphism we have the following identifications

$$\xi \mapsto \zeta, \quad u_1 \mapsto \zeta + h, \quad u_2 \mapsto \zeta h.$$

Proof. For the first point, in Proposition 4.4 we have seen that $\widetilde{\Omega}$ is the relative Lagrangian Grassmannian of $T\Omega$. Let \mathcal{U} be the tautological subbundle, then, since it is Lagrangian, we have the short exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow T\Omega \rightarrow \mathcal{U}^\vee \rightarrow 0.$$

From this one gets the relations for the Chern classes of \mathcal{U} . To see that these are all the relations between the Chern classes of \mathcal{U} , we consider the Leray spectral sequence for $p_\Omega: \widetilde{\Omega} \rightarrow \Omega$:

$$E_2^{p,q} = H^p(R^q p_* \underline{\mathbb{Q}}_{\widetilde{\Omega}}) \Rightarrow H^{p+q}(\widetilde{\Omega}).$$

As Ω is simply connected, the local system $R^q p_* \underline{\mathbb{Q}}_{\widetilde{\Omega}}$ is trivial. Thus the Chern classes of \mathcal{U} generate the cohomology and we get

$$H^k(\widetilde{\Omega}) \simeq \bigoplus_{p+q=k} H^p(\Omega) \otimes H^q(G),$$

where $G := LG(V)$ is the Lagrangian Grassmannian of a four dimensional space V . Finally comparing the dimensions we conclude that the relations we found are the only ones they satisfy.

The other points are now just an application of the projective bundle formula. \square

4.5. The cohomology of the Σ -varieties. We call $\mathcal{L}, \mathcal{U}, \mathcal{F}$ the universal subbundles on $\mathbb{P}_\Omega(T\Omega) \simeq \overline{\Omega}$, $\mathcal{L}\mathcal{G}_\Omega(T\Omega) \simeq \widetilde{\Omega}$, $\mathbb{P}_\Omega(\mathcal{L}^\perp/\mathcal{L}) \simeq \widehat{\Omega}$ respectively in the case $Y = M$ and on $\mathbb{P}_\Omega(V) \simeq \overline{\Omega}$, $LG_\Omega(V) \simeq \widetilde{\Omega}$, $\mathbb{P}_\Omega(\mathcal{L}^\perp/\mathcal{L}) \simeq \widehat{\Omega}$ respectively in the case $Y = K$, as introduced in the previous section.

Proposition 4.6. *Let Y be the variety M or K .*

- (1) Under the isomorphism $\widehat{\Omega} \simeq \mathbb{P}_{\widehat{\Omega}}(\mathcal{U})$ we have $c_1(\mathcal{O}_{\widehat{\Sigma}}(\widehat{\Omega})|_{\widehat{\Omega}}) = c_1(\mathcal{O}_{\widehat{\Omega}}(\widehat{\Omega}_{OG})) = 2c_1(\mathcal{L})$.
- (2) Under the isomorphism $\widehat{\Omega} \simeq \mathbb{P}_{\widehat{\Omega}}(\mathcal{L}^\perp/\mathcal{L})$ we have $c_1(\mathcal{O}_{\widehat{\Omega}}(\widehat{\Sigma})) = 2c_1(\mathcal{F}) - 2c_1(\mathcal{L})$.
- (3) Under the isomorphism $\overline{\Omega} \simeq \mathbb{P}_{\overline{\Omega}}(T\Omega)$ for $Y = M$ and $\overline{\Omega} \simeq \mathbb{P}_{\overline{\Omega}}(V)$ for $Y = K$ we have $c_1(\mathcal{O}_{\overline{\Sigma}}(\overline{\Omega})|_{\overline{\Omega}}) = 2c_1(\mathcal{L})$.
- (4) Under the isomorphism $\widetilde{\Omega} \simeq \mathcal{L}\mathcal{G}_{\Omega}(T\Omega)$ for $Y = M$ and $\widetilde{\Omega} \simeq LG_{\Omega}(V)$ for $Y = K$ we have $c_1(\mathcal{O}_{\widetilde{\Omega}}(\widetilde{\Sigma})) = 2c_1(\mathcal{U})$.

Proof. We prove the result for $Y = M$. When $Y = K$ the same proof applies, and O'Grady's results used in item (1) and (2) hold true, cfr. [19, §2.1].

- (1) The first equality holds by construction, as $\widehat{\Omega} = \widehat{\Omega}_{OG} \cap \widehat{\Sigma}$. The second equality follows from [18, (2.0.1) Proposition, item (2)] and the fact that the embedding $\widehat{\Omega} \subset \widehat{\Omega}_{OG}$ is identified with $\mathbb{P}_{\widehat{\Omega}}(\mathcal{U}) \subset \mathbb{P}_{\widehat{\Omega}}(S^2\mathcal{U})$ by construction.
- (2) From [18, Equation (2.2.1)] $\omega_{\widehat{M}} = \mathcal{O}_{\widehat{M}}(2\widehat{\Omega}_{OG})$. By adjunction for the inclusions $\widehat{\Omega} \subset \widehat{\Sigma}$ and $\widehat{\Sigma} \subset \widehat{M}$ we compute

$$\omega_{\widehat{\Omega}} = (\omega_{\widehat{\Sigma}} \otimes \mathcal{O}_{\widehat{\Sigma}}(\widehat{\Omega}))|_{\widehat{\Omega}} = \left((\omega_{\widehat{M}} \otimes \mathcal{O}_{\widehat{M}}(\widehat{\Sigma}))|_{\widehat{\Sigma}} \otimes \mathcal{O}_{\widehat{\Sigma}}(\widehat{\Omega}) \right)|_{\widehat{\Omega}} = \mathcal{O}_{\widehat{\Sigma}}(3\widehat{\Omega}_{OG} + \widehat{\Sigma})|_{\widehat{\Omega}}$$

hence $c_1(\mathcal{O}_{\widehat{\Omega}}(\widehat{\Sigma})) = c_1(\omega_{\widehat{\Omega}}) - 6c_1(\mathcal{L})$ thanks to part (1). On the other hand, $\widehat{\Omega}$ and $\overline{\Omega}$ are projective bundles on $\overline{\Omega}$ and Ω respectively, hence using the Euler sequence we can express

$$\begin{aligned} \omega_{\widehat{\Omega}} &= g_{\Omega}^* \omega_{\overline{\Omega}} \otimes \det(\mathcal{L}^\perp/\mathcal{L})^\vee \otimes \mathcal{F}^{\otimes 2} = g_{\Omega}^* \omega_{\overline{\Omega}} \otimes \det(L^\perp/L)^\vee \otimes \mathcal{F}^{\otimes 2} = \\ &= g_{\Omega}^* q_{\Omega}^* \omega_{\Omega} \otimes \det T\Omega^\vee \otimes \mathcal{L}^{\otimes 4} \otimes \det(\mathcal{L}^\perp/\mathcal{L})^\vee \otimes \mathcal{F}^{\otimes 2} = \mathcal{L}^{\otimes 4} \otimes \mathcal{F}^{\otimes 2} \end{aligned}$$

where $\det(\mathcal{L}^\perp/\mathcal{L}) \cong \mathcal{O}_{\widehat{\Omega}}$ has been computed in (4.1). Comparing the two writings we get $c_1(\mathcal{O}_{\widehat{\Omega}}(\widehat{\Sigma})) = 2c_1(\mathcal{F}) - 2c_1(\mathcal{L})$.

- (3) Consider the pullback

$$f_{\Omega}^* c_1(\mathcal{O}_{\overline{\Sigma}}(\overline{\Omega})|_{\overline{\Omega}}) = c_1(\mathcal{O}_{\widehat{\Sigma}}(\widehat{\Omega})|_{\widehat{\Omega}}) = 2c_1(\mathcal{L})$$

where the last equality is part (1). The claim now follows from the injectivity of f^* .

- (4) There exists some $\lambda_1, \lambda_2 \in \mathbb{R}$ and some $\omega \in H^2(\Omega)$ such that

$$\begin{aligned} g_{\Omega}^*(\lambda_1 c_1(\mathcal{U}) + \omega) &= g_{\Omega}^* c_1(\mathcal{O}_{\widetilde{\Omega}}(\widetilde{\Sigma})) = g_{\Omega}^* \widetilde{i}_{\Omega}^* c_1(\mathcal{O}_{\widetilde{M}}(\widetilde{\Sigma})) \\ &= \widehat{i}_{\Omega}^* \gamma^* c_1(\mathcal{O}_{\widehat{M}}(\widetilde{\Sigma})) = \widehat{i}_{\Omega}^* c_1(\mathcal{O}_{\widehat{M}}(\widehat{\Sigma} + \lambda_2 \widehat{\Omega}_{10})) \\ &= 2c_1(\mathcal{F}) + 2(\lambda_2 - 1)c_1(\mathcal{L}). \end{aligned}$$

As $H^2(\widehat{\Omega}) = H^2(\Omega) \oplus \mathbb{Q}c_1(\mathcal{U}) \oplus \mathbb{Q}c_1(\mathcal{L})$, and $c_1(\mathcal{U}) = c_1(\mathcal{F}) + c_1(\mathcal{L})$, we have $\lambda_1 = 2, \omega = 0, \lambda_2 = 2$. Using the injectivity of g_{Ω}^* we get the claim. \square

Remark 4.7. We extract from the proof of item (4) here above the following relation, that we are going to use in what follows:

$$\gamma^* \mathcal{O}_{\widehat{M}}(\widetilde{\Sigma}) = \mathcal{O}_{\widehat{M}}(\widehat{\Sigma} + 2\widehat{\Omega}_{OG}).$$

Proposition 4.8. *Let Y be the variety M or K . The cohomology of the Σ -varieties is described as follows.*

(1) *The pullback along the morphism $q: \bar{\Sigma} \rightarrow \Sigma$ is injective. Let $\zeta = c_1(\mathcal{L}) \in H^2(\bar{\Omega})$. For any k we have*

$$H^k(\bar{\Sigma}) = q^* H^k(\Sigma) \oplus \bar{i}_* q_{\Omega}^* H^{k-2}(\Omega) \oplus \bar{i}_* \zeta q_{\Omega}^* H^{k-4}(\Omega) \oplus \bar{i}_* \zeta^2 q_{\Omega}^* H^{k-6}(\Omega).$$

(2) *The pullback along the morphism $f: \widehat{\Sigma} \rightarrow \bar{\Sigma}$ is injective, the multiplication $H^{k-2}(\widehat{\Sigma}) \rightarrow H^k(\widehat{\Sigma})$ with $c_1(\mathcal{O}_{\widehat{\Sigma}}(-\widehat{\Sigma}))$ is injective on $f^* H^{k-2}(\bar{\Sigma})$ for any k . Moreover, for any k we have*

$$H^k(\widehat{\Sigma}) = f^* H^k(\bar{\Sigma}) \oplus c_1(\mathcal{O}_{\widehat{\Sigma}}(-\widehat{\Sigma})) \cdot f^* H^{k-2}(\bar{\Sigma}).$$

(3) *The pullback along the morphism $g: \widetilde{\Sigma} \rightarrow \widehat{\Sigma}$ is injective and for any k we have*

$$H^k(\widetilde{\Sigma}) = g^* H^k(\widehat{\Sigma}) \oplus \widehat{i}_* q_{\Omega}^* H^{k-2}(\widetilde{\Omega}).$$

(4) *The pullback along the morphism $p: \widetilde{\Sigma} \rightarrow \Sigma$ is injective.*

Proof. (1) Since Σ has quotient singularities, the cohomology groups $H^k(\Sigma)$ carry a pure Hodge structure of weight k and the pullback $H^k(\Sigma) \rightarrow H^k(\bar{\Sigma})$ is injective [23]. The diagram

$$\begin{array}{ccc} \bar{\Sigma} & \longleftarrow & \bar{\Omega} \\ \downarrow q & \bar{i} & \downarrow q_{\Omega} \\ \Sigma & \longleftarrow & \Omega \\ & i & \end{array}$$

is a hypercubical resolution of Σ and its weight spectral sequence, which degenerates at the second page, is supported on the first two columns. As the cohomology of Σ carries a pure Hodge structure the differential at the first page must be surjective and we get this short exact sequence

$$(4.2) \quad 0 \rightarrow H^k(\Sigma) \xrightarrow{(q^*, i^*)} H^k(\bar{\Sigma}) \oplus H^k(\Omega) \xrightarrow{\bar{i}^* - q_{\Omega}^*} H^k(\bar{\Omega}) \rightarrow 0.$$

We can split the sequence by defining the following section

$$\begin{aligned} \bigoplus_{r=0}^3 \zeta^r q^* H^{k-2r}(\Omega) &= H^k(\bar{\Omega}) \rightarrow H^k(\bar{\Sigma}) \oplus H^k(\Omega) \\ (\zeta^r q_{\Omega}^* a_r)_r &\mapsto \left(\frac{1}{2} (\bar{i}_* \zeta^{r-1} q_{\Omega}^* a_r)_{r \geq 1}, -a_0 \right). \end{aligned}$$

To see that it is a section, we notice that

$$\bar{i}^* \bar{i}_* \frac{1}{2} \zeta^{r-1} q_{\Omega}^* a_r = \frac{1}{2} [\bar{\Omega}]|_{\bar{\Omega}} \cdot \zeta^{r-1} q_{\Omega}^* a_r = \zeta^r q_{\Omega}^* a_r$$

where the first equality follows from [26, Chapter 11, Exercise 1] and the last one from (3) in Proposition 4.6.

From the short exact sequence (4.2) we can then extract the desired decomposition.

- (2) As the morphism $\widehat{\Sigma} \rightarrow \overline{\Sigma}$ is a \mathbb{P}^1 -fibration and $c_1(\mathcal{O}_{\widehat{\Sigma}}(-\widehat{\Sigma}))$ pairs with the generic fiber in 2 points by [18, Proposition (2.3.1)], the claim follows from the projective bundle formula.
- (3) The proof is completely similar to point (1).
- (4) Consider the diagram

$$\begin{array}{ccc}
 & \widehat{\Sigma} & \\
 f \swarrow & & \searrow g \\
 \widetilde{\Sigma} & & \overline{\Sigma} \\
 p \searrow & & \swarrow q \\
 & \Sigma &
 \end{array}$$

We have already seen that the pullbacks along q, g and f are injective, hence the pullback along p is too. □

We conclude describing the object $E_2^{0,k}$ in the spectral sequence (3.6).

Proposition 4.9. *We have*

$$\begin{aligned}
 E_2^{0,k} &\simeq W_k := \{y \in H^k(\widetilde{Y}) : \widetilde{j}^* m = p^* \sigma' \text{ for some } \sigma' \in H^k(\Sigma)\} \\
 (y, \sigma, \omega) &\longmapsto y.
 \end{aligned}$$

Proof. We write

$$(y, \sigma, \omega) \in H^k(\widetilde{Y}) \oplus H^k(\overline{\Sigma}) \oplus H^k(\Omega) = E_1^{0,k}$$

for an element in $E_1^{0,k}$ and we have the following

Claim. $\ker(d_0 : E_1^{0,k} \rightarrow E_1^{1,k})$ consist of elements $(y, \sigma, \omega) \in E_1^{0,k}$ such that:

- (1) $\sigma = q^* \sigma'$ for some $\sigma' \in H^k(\Sigma)$ with $i^* \sigma' = \omega$.
- (2) $\widetilde{j}^* y = p^* \sigma'$.

Proof of the claim. We want to check the three conditions in Lemma 3.2.

- (1) For any element (y, σ, ω) of the kernel we have that $\bar{i}^* \sigma = q_{\overline{\Omega}}^* \omega \in H^k(\overline{\Omega})$. By Proposition 4.6 the restriction map \bar{i}^* respects the decomposition of $H^k(\overline{\Sigma})$ of Proposition 4.8.(1) and of $H^k(\overline{\Omega})$ of Proposition 4.3.(2) and Proposition 4.4.(2); hence the condition above translates to $\sigma = q^* \sigma'$ for some $\sigma' \in H^k(\Sigma)$ such that $i^* \sigma' = \omega$.
- (2) For any element of the kernel $(y, q^* \sigma', \omega)$, we have that the pullback of y and $q^* \sigma'$ in $H^k(\widehat{\Sigma})$ coincide, thus

$$g^* \widetilde{j}^* y = f^* q^* \sigma' = g^* p^* \sigma'$$

and point (2) follows from the injectivity of g^* . □

It follows that any element in $\ker(d_0 : E_1^{0,k} \rightarrow E_1^{1,k}) = E_2^{0,k}$ is determined by the choice of $y \in H^k(\tilde{Y})$, as p^* is injective (because $g^*p^* = f^*q^*$ is injective) and $\omega = i^*\sigma'$. Such $y \in H^k(\tilde{Y})$ needs to satisfy $\tilde{j}^*m = p^*\sigma'$ and the claim follows. \square

5. THE COHOMOLOGY OF THE 10-DIMENSIONAL SINGULAR MODULI SPACE

Assume the notations as in Notation 2.13. In this section we discuss the case of the 10-dimensional singular O'Grady's moduli space $Y = M$. Using the results of the previous section we can compute the Betti numbers of the Ω - and Σ -varieties, that we list in the table below.

As observed at the beginning of Section 2 the manifold Ω is a Hyperkähler manifold of K3^[2]-type and its Betti numbers are known thanks to the Göttsche formula [8]. Thanks to Proposition 4.4 a straightforward computation gives the Betti numbers of the other Ω -varieties.

Since the variety Σ is isomorphic to double symmetric product of Ω , the rational cohomology of Σ is isomorphic to the invariant part of the rational cohomology $\Omega \times \Omega$ [3, §III, Theorem 2.4], thus we can compute its Betti numbers. Using Proposition 4.8 we can compute all Betti numbers of the other Σ -varieties.

Finally, the cohomology of the manifold \tilde{M} has been computed in [5].

	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}
Ω	1	23	276	23	1	0	0	0	0	0	0
$\bar{\Omega}$	1	24	300	323	323	300	24	1	0	0	0
$\tilde{\Omega}$	1	24	300	323	323	300	24	1	0	0	0
$\hat{\Omega}$	1	25	324	623	646	623	324	25	1	0	0
Σ	1	23	552	6371	38756	6371	552	23	1	0	0
$\bar{\Sigma}$	1	24	576	6671	39078	6671	576	24	1	0	0
$\tilde{\Sigma}$	1	24	576	6947	45426	45426	6947	576	24	1	0
$\hat{\Sigma}$	1	25	600	7247	45749	45749	7247	600	25	1	0
\tilde{M}	1	24	300	2899	22150	126156	22150	2899	300	24	1

Corollary 5.1. *The Euler characteristic of M is $\chi(M) = 123606$.*

Proof. It follows immediately from the dimensions computed in (5.1), as $\chi(M) = \chi(\tilde{M}) - \chi(\tilde{\Sigma}) + \chi(\Sigma)$. \square

Proposition 5.2. *We have, for any $k \geq 0$:*

- (1) $i^* : H^k(\Sigma) \rightarrow H^k(\Omega)$ is surjective.
- (2) $\bar{i}^* : H^k(\bar{\Sigma}) \rightarrow H^k(\bar{\Omega})$ is surjective.
- (3) $\hat{i}^* : H^k(\hat{\Sigma}) \rightarrow H^k(\hat{\Omega})$ is surjective.
- (4) $j^* : H^2(M) \rightarrow H^2(\Sigma)$ is an isomorphism.
- (5) $\tilde{i}_M^* : H^k(\tilde{M}) \rightarrow H^k(\tilde{\Omega})$ is surjective.

Remark 5.3. The variety M has a symplectic form on its regular part, in other words a reflexive form $\sigma_M \in H^0(\Omega_M^{[2]}, M)$, where $\Omega_M^{[2]} = (\Omega_M^2)^{\vee\vee} = (i_*^{reg} \Omega_{M^{reg}}^2)$. This induces a nontrivial class in $H^2(M, \mathbb{C})$. An analogous statement holds for Σ .

Proof. Since the involved varieties have no odd cohomology (cfr. (5.1)) we only need to prove the statement for their even cohomology groups.

- (1) For $k = 0$ and $k > 8$ the statement is obvious. For $k = 2$: given the transcendental lattice $T(\Sigma) \subset H^2(\Sigma, \mathbb{Z})$, the intersection $\ker(i^*) \cap T(\Sigma) \subseteq T(\Sigma)$ is a Hodge substructure. The reflexive symplectic form of Σ restricts to the symplectic form of Ω ([11, Theorem 2.3]), hence the orthogonal complement of $\ker(i^*) \cap T(\Sigma)$ in $T(\Sigma)$ contains the symplectic form of Σ and then it coincides with $T(\Sigma)$ by minimality of the transcendental lattice. It follows that i^* is injective on $T(\Sigma)$. Taking a general locally trivial deformation Σ_t of Σ and defining Ω_t its singular locus, for the very general Σ_t we have that $H^2(\Sigma_t, \mathbb{Z}) = T(\Sigma_t)$, hence the pullback of the inclusion $i_t : \Omega_t \hookrightarrow \Sigma_t$ is injective on $H^2(\Sigma_t, \mathbb{Z})$; we conclude that $i^* : H^2(\Sigma) \rightarrow H^2(\Omega)$ is injective, hence an isomorphism for dimensional reasons, cfr. (5.1). For the case $k = 4$ we consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^2 H^2(\Sigma) & \xrightarrow{\mathrm{Sym}^2 i^*} & \mathrm{Sym}^2 H^2(\Omega) \\ \downarrow & & \downarrow \\ H^4(\Sigma) & \xrightarrow{i^*} & H^4(\Omega). \end{array}$$

The left and right vertical arrow are injective by [1, Proposition 5.16] and [25, Theorem 1.5] respectively, hence the right one is an isomorphism for dimensional reasons and i^* is surjective. We are left with the cases $k = 6, 8$. For the case $k = 6$, let $L : H^k(\Sigma) \rightarrow H^{k+2}(\Sigma)$ be the Lefschetz operator and L_Ω its restriction to Ω . We have the commutative square:

$$\begin{array}{ccc} H^2(\Sigma) & \xrightarrow{i^*} & H^2(\Omega) \\ \downarrow L^2 & & \downarrow L_\Omega^2 \\ H^6(\Sigma) & \xrightarrow{i^*} & H^6(\Omega) \end{array}$$

The right vertical arrow is an isomorphism by Hard Lefschetz, hence the surjectivity of i^* in degree 6 follows from the surjectivity of i^* in degree 2. The case $k = 8$ is proved with the same argument, starting from the surjectivity of $i^* : H^0(\Sigma) \rightarrow H^0(\Omega)$.

- (2) Following Proposition 4.8 and Proposition 4.4:

$$\begin{aligned} H^k(\overline{\Sigma}) &= q^* H^k(\Sigma) \oplus \bar{i}_* [q_\Omega^* H^{k-2}(\Omega) \oplus \zeta \cdot q_\Omega^* H^{k-4}(\Omega) \oplus \zeta^2 \cdot q_\Omega^* H^{k-6}(\Omega)], \\ H^k(\overline{\Omega}) &= q_\Omega^* H^k(\Omega) \oplus \zeta \cdot q_\Omega^* H^{k-2}(\Omega) \oplus \zeta^2 \cdot q_\Omega^* H^{k-4}(\Omega) \oplus \zeta^3 \cdot q_\Omega^* H^{k-6}(\Omega). \end{aligned}$$

We look at the morphism $\bar{i}^* : H^k(\bar{\Sigma}) \rightarrow H^k(\bar{\Omega})$ on the factors of the above decomposition. From the surjectivity of i^* proved in (1) it follows:

$$\bar{i}^* q^* H^k(\Sigma) = q_\Omega^* i^* H^k(\Sigma) = q_\Omega^* H^k(\Omega).$$

Furthermore, from Proposition 4.6:

$$\bar{i}^* \bar{i}_* q_\Omega^* H^l(\Omega) = [\bar{\Omega}]|_{\bar{\Omega}} \cdot q_\Omega^* H^l(\Omega) = 2\zeta \cdot q_\Omega^* H^l(\Omega)$$

hence \bar{i}^* is surjective for any $k \geq 0$.

(3) Following Proposition 4.4.(3) and Proposition 4.8.(2) we have

$$H^k(\widehat{\Sigma}) = f^* H^k(\bar{\Sigma}) \oplus c_1(\mathcal{O}_{\widehat{\Sigma}}(-\widehat{\Sigma})) \cdot f^* H^{k-2}(\bar{\Sigma})$$

$$H^k(\widehat{\Omega}) = f_\Omega^* H^k(\bar{\Omega}) \oplus h \cdot f_\Omega^* H^{k-2}(\bar{\Omega}).$$

Using the decompositions above: from the surjectivity in part (2) we have that $\widehat{i}^*|_{f^* H^k(\bar{\Sigma})} : f^* H^k(\bar{\Sigma}) \rightarrow f_\Omega^* \bar{i}^* H^k(\bar{\Sigma}) = f_\Omega^* H^k(\bar{\Omega})$, hence the claim follows from 4.6.(2).

(4) The injectivity of j^* is proven exactly as the injectivity of $i^* : H^2(\Sigma) \rightarrow H^2(\Omega)$ in item (1), hence we conclude for dimensional reasons, cfr. [21, Theorem 1.7] and (5.1).

(5) As Ω is of $K3^{[2]}$ -type, it is well-known that $H^*(\Omega) = H^0(\Omega)\langle H^2(\Omega) \rangle$, cfr. [9, Corollary 3.2]. Proposition 4.4 shows that

$$H^*(\widetilde{\Omega}) = H^0(\widetilde{\Omega})\langle p_\Omega^* H^2(\Omega), u_1 \rangle = \langle H^2(\widetilde{\Omega}) \rangle.$$

We have

$$\widetilde{i}_M^* H^2(\widetilde{M}) = \widetilde{i}_M^* (\pi^* H^2(M) \oplus c_1(\mathcal{O}_{\widetilde{M}}(\widetilde{\Sigma})) \cdot \mathbb{Q}) = p_\Omega^* H^2(\Omega) \oplus u_1 \cdot \mathbb{Q} = H^2(\widetilde{\Omega})$$

where the second equality follows from the surjectivity of $i_M^* : H^2(M) \rightarrow H^2(\Omega)$, obtained as combination of item (1) and item (4), and Proposition 4.6.(4).

We conclude that the pullback \widetilde{i}_M^* is surjective in any degree. □

Proposition 5.4. *We have*

$$H^{2k}(M) \simeq E_2^{0,2k} \quad \text{and} \quad H^{2k+1}(M) \simeq E_2^{1,2k}.$$

In particular, if non-zero the groups $H^{2k}(M)$ and $H^{2k+1}(M)$ carry a pure Hodge structure of weight $2k$.

Proof. At first, notice that $E_2^{2,k} = 0$ for any $k \in \mathbb{Z}$: the differential $d_1 : E_1^{1,k} \rightarrow E_1^{2,k}$ is surjective by Lemma 3.3 and Proposition 5.2.(3). Hence the statement follows from the fact that all varieties appearing in the semi-simplicial resolution of M have trivial cohomology in odd degree. □

Corollary 5.5. *The pullback $\pi^* : H^{2k}(M) \rightarrow H^{2k}(\widetilde{M})$ is injective for any k .*

Proof. Thanks to the above proposition, the morphism

$$(5.2) \quad \pi^* \oplus q^* i^* \oplus i_M^*: H^{2k}(M) \rightarrow H^{2k}(\widetilde{M}) \oplus H^{2k}(\overline{\Sigma}) \oplus H^{2k}(\Omega)$$

is an isomorphism on its image $E_2^{0,2k}$. By Proposition 4.9 the restriction of the projection onto the cohomology of \widetilde{M}

$$(5.3) \quad E_2^{0,2k} \subset H^{2k}(\widetilde{M}) \oplus H^{2k}(\overline{\Sigma}) \oplus H^{2k}(\Omega) \rightarrow H^{2k}(\widetilde{M})$$

is an isomorphism onto its image. Taking the composition of (5.2) and (5.3) we get the claim. \square

Proposition 5.6. *We denote $e_k^{i,j} = \dim E_k^{i,j}$. We have*

$$\begin{aligned} b_{2k}(M) &\leq b_{2k}(\widetilde{M}) - (b_{2k}(\widetilde{\Omega}) - b_{2k}(\Omega)), \\ b_{2k}(M) &\geq e_1^{0,2k} - e_1^{1,2k} + e_1^{2,2k} \quad \text{and} \\ b_{2k+1}(M) &= e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(M). \end{aligned}$$

Proof. Recall the definition of W_{2k} from Proposition 4.9 and that $\dim W_{2k} = b_{2k}(M)$ from Proposition 4.9. We estimate the dimension of W_{2k} . As $H^{2k}(\widetilde{M}) \rightarrow H^{2k}(\widetilde{\Omega})$ is surjective from item (4) of Proposition 5.2 and under this map W_k maps onto $p_{\widetilde{\Omega}}^* H^{2k}(\Omega)$ we get a surjective map on the quotients

$$H^{2k}(\widetilde{M})/W_{2k} \rightarrow H^{2k}(\widetilde{\Omega})/p_{\widetilde{\Omega}}^* H^{2k}(\Omega)$$

whence the first claimed inequality.

For the second inequality, we look at the complex

$$E_1^{0,2k} \xrightarrow{d_0} E_1^{1,2k} \xrightarrow{d_1} E_1^{2,2k} \rightarrow 0$$

then using Proposition 5.4 we obtain

$$b_{2k}(M) = \dim \ker d_0 = e_1^{0,2k} - \dim \operatorname{Im} d_0 \geq e_1^{0,2k} - \dim \ker d_1 = e_1^{0,2k} - e_1^{1,2k} + e_1^{2,2k}$$

where the last equality follows from the surjectivity of d_1 , cfr. Lemma 3.3 and Proposition 5.2.(3).

For the last inequality, as $H^{2k+1}(M) \simeq \ker d_1 / \operatorname{Im} d_0$ (cfr. Proposition 5.4) we have

$$b_{2k+1}(M) = \dim \ker d_1 - \dim \operatorname{Im} d_0 = e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(M)$$

where $\dim \ker d_1 = e_1^{1,2k} - e_1^{2,2k}$ follows from the surjectivity of d_1 , cfr. Lemma 3.3 and Proposition 5.2.(3). \square

Remark 5.7. Using the relation $b_{2k+1}(M) = e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(M)$ computed in Proposition 5.6 we get back the Euler characteristic as computed in Corollary 5.1:

$$\begin{aligned} \chi(M) &= \sum_{k=0}^{10} (b_{2k}(M) - b_{2k+1}(M)) = \sum_{k=0}^{10} (e_1^{0,2k} - e_1^{1,2k} + e_1^{2,2k}) \\ &= \chi(\widetilde{M}) + \chi(\Omega) + \chi(\widetilde{\Sigma}) - \chi(\widehat{\Sigma}) - \chi(\widetilde{\Omega}) - \chi(\overline{\Omega}) + \chi(\widehat{\Omega}) \\ &= \chi(\widetilde{M}) - \chi(\widetilde{\Sigma}) + \chi(\Sigma) \\ &= 123606 \end{aligned}$$

as $\chi(\widetilde{\Sigma}) = \chi(\widehat{\Sigma}) - \chi(\widehat{\Omega}) + \chi(\widetilde{\Omega})$ and $\chi(\Sigma) = \chi(\overline{\Sigma}) - \chi(\overline{\Omega}) + \chi(\Omega)$. In particular, the knowledge of the Euler characteristic of M does not give any further information on its Betti numbers.

Remark 5.8. For $2k \leq 10$ we have an injection $\text{Sym}^k(M) \hookrightarrow H^{2k}(M)$ by [1, Proposition 5.16], which gives $b_{2k}(M) \geq \binom{22+k}{k}$. Nevertheless, this estimate turns out to be weaker than (or equal to) the one in Proposition 5.6.

Corollary 5.9. *We call $b_i := b_i(M)$. We have*

$b_0 = 1$	$b_1 = 0$	$b_2 = 23$
$b_3 = 0$	$b_4 = 276$	$b_5 = 0$
$2323 \leq b_6 \leq 2599$	$b_7 \leq 276$	$15480 \leq b_8 \leq 21828$
$b_9 \leq 6348$	$87101 \leq b_{10} \leq 125856$	$b_{11} \leq 38755$
$15755 \leq b_{12} \leq 22126$	$b_{13} \leq 6371$	$2346 \leq b_{14} \leq 2898$
$b_{15} \leq 552$	$277 \leq b_{16} \leq 300$	$b_{17} \leq 23$
$23 \leq b_{18} \leq 24$	$b_{19} \leq 1$	$b_{20} = 1.$

Proof. This is a direct computation of the bounds in Proposition 5.6 using the dimensions in (5.1). The estimates on odd Betti numbers are given by the following formula:

$$b_{2k+1}(M) \leq e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(\widetilde{M}) - (b_{2k}(\widetilde{\Omega}) - b_{2k}(\Omega))$$

which is straightforward from Proposition 5.6. \square

Remark 5.10. The 1st and 2nd Betti numbers were already known [21, Theorem 1.7] and [2, Lemma 2.1].

Proposition 5.11. *We have*

$$\begin{aligned} g^* p^* \text{Sym}^{9-k} H^2(\Sigma) &\subseteq \text{Im}\{d_0: E_1^{0,2k} \rightarrow E_1^{1,2k}\} \quad \text{for } 10 \leq 2k \leq 12, \\ g^* \text{Sym}^{9-k} H^2(\widetilde{\Sigma}) &\subseteq \text{Im}\{d_0: E_1^{0,2k} \rightarrow E_1^{1,2k}\} \quad \text{for } 14 \leq 2k \leq 18. \end{aligned}$$

As consequence, if we denote $e_k^{i,j} = \dim E_k^{i,j}$ then we have

$$\begin{aligned} \text{for } 10 \leq 2k \leq 12 : b_{2k}(M) &\leq e_1^{0,2k} - \binom{31-k}{9-k} \\ b_{2k+1}(M) &\leq e_1^{1,2k} - e_1^{2,2k} - \binom{31-k}{9-k} \\ \text{for } 14 \leq 2k \leq 18 : b_{2k}(M) &\leq e_1^{0,2k} - \binom{32-k}{9-k} \\ b_{2k+1}(M) &\leq e_1^{1,2k} - e_1^{2,2k} - \binom{32-k}{9-k}. \end{aligned}$$

Proof. Assume $10 \leq 2k \leq 18$. From the very definition of the spectral sequence (3.6) we have $(\tilde{j} \circ g)^* H^{2k}(\tilde{M}) \subseteq \text{Im}\{d_0 : E_1^{0,2k} \rightarrow E_1^{1,2k}\}$. Consider the Lefschetz operator $L : H^k(\tilde{M}) \rightarrow H^{k+2}(\tilde{M})$ and its restriction $L_{\tilde{\Sigma}}$ to $\tilde{\Sigma}$. We have a commutative square

$$\begin{array}{ccc} H^{2k}(\tilde{M}) & \xrightarrow{\tilde{j}^*} & H^{2k}(\tilde{\Sigma}) \\ L^{2k-10} \uparrow \simeq & & L_{\tilde{\Sigma}}^{2k-10} \uparrow \\ H^{2(10-k)}(\tilde{M}) & \xrightarrow{\tilde{j}^*} & H^{2(10-k)}(\tilde{\Sigma}) \simeq \\ L \uparrow & & L \uparrow \\ H^{2(9-k)}(\tilde{M}) & \xrightarrow{\tilde{j}^*} & H^{2(9-k)}(\tilde{\Sigma}). \end{array}$$

where the isomorphism $H^{2(9-k)}(\tilde{\Sigma}) \simeq H^{2k}(\tilde{\Sigma})$ is given by Hard Lefschetz for intersection cohomology [4, §1.4]. Furthermore we consider the following commutative diagram

$$\begin{array}{ccc} H^{2(9-k)}(\tilde{M}) & \xrightarrow{\tilde{j}^*} & H^{2(9-k)}(\tilde{\Sigma}) \\ \pi^* \uparrow & & p^* \uparrow \\ H^{2(9-k)}(M) & \xrightarrow{j^*} & H^{2(9-k)}(\Sigma) \\ \uparrow & & \uparrow \\ \text{Sym}^{9-k} H^2(M) & \xrightarrow[\sim]{\text{Sym} j^*} & \text{Sym}^{9-k} H^2(\Sigma) \end{array}$$

where: π^* is injective by Corollary 5.5, p^* is injective by Proposition 4.8.(4), the vertical maps from the symmetric products to the respective cohomology groups are injective because by assumption $9-k \leq 4$, see [1, Proposition 5.16], and $\text{Sym}^{9-k} j^*$ is an isomorphism by Proposition 5.2.(4). We conclude that $p^* \text{Sym}^{9-k} H^2(\Sigma) \subseteq \tilde{j}^* H^{2k}(\tilde{M})$, hence the statement.

When $14 \leq 2k \leq 18$ we can improve the result as follows. Consider the following commutative diagram

$$(5.4) \quad \begin{array}{ccccc} H^{2(9-k)}(\widetilde{M}) & \xrightarrow{\widetilde{j}^*} & H^{2(9-k)}(\widetilde{\Sigma}) & \xrightarrow{\widetilde{i}^*} & H^{2(9-k)}(\widetilde{\Omega}) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Sym}^{9-k} H^2(\widetilde{M}) & \xrightarrow{\widetilde{\text{Sym}} \widetilde{j}^*} & \text{Sym}^{9-k} H^2(\widetilde{\Sigma}) & \xrightarrow{\widetilde{\text{Sym}} \widetilde{i}^*} & \text{Sym}^{9-k} H^2(\widetilde{\Omega}) \end{array}$$

where the left vertical arrow is injective because by hypothesis $9 - k \leq 2$, see [25, Theorem 1.5], and the lower horizontal arrows are isomorphisms by Proposition 5.2.(5) and dimensional reasons. Using the description of $H^2(\widetilde{\Omega})$ in Proposition 4.4.(1) we get

$$\text{Sym}^{9-k} H^2(\widetilde{\Omega}) = \text{Sym}^{9-k} p_{\Omega}^* H^2(\Omega) \bigoplus_{l=1}^{9-k} u_1^l \cdot \text{Sym}^{9-k-l} p_{\Omega}^* H^2(\Omega)$$

where we are assuming as convention $p_{\Omega}^* \text{Sym}^0 H^2(\Omega) = p_{\Omega}^* H^0(\Omega)$; note that in the direct sum above $l \leq 2$. Using the injectivity of $\text{Sym}^m H^2(\Omega) \rightarrow H^{2m}(\Omega)$ for any $m \leq 2$, cfr. again [25, Theorem 1.5], and the decomposition

$$H^{2(9-k)}(\widetilde{\Omega}) = p_{\Omega}^* H^{2(9-k)}(\Omega) \bigoplus_{l=1}^{9-k} u_1^l \cdot p_{\Omega}^* H^{2(9-k-l)}(\Omega)$$

given by Proposition 4.4.(1) we obtain that the right vertical arrow in the diagram (5.4) is injective, hence the same holds true for the right central arrow of the diagram. We conclude that $\text{Sym}^{9-k} H^2(\widetilde{\Sigma}) \subseteq \widetilde{j}^* H^{2k}(\widetilde{M})$ hence the first statement.

The claim on the Betti numbers is obtained as in the proof of Proposition 5.6, using again the injectivity of p^* and the one of g^* , cfr. Proposition 4.8. \square

Remark 5.12. Notice that crucial in the argument of Proposition 5.11 is the study of the kernel $H^2(\widetilde{M}) \rightarrow H^2(\widetilde{\Sigma})$, which is a Hodge substructure of $H^2(\widetilde{M})$. Now, the cohomology ring $H^*(\widetilde{M})$ decomposes in irreducible representations for its LLV algebra and in the argument we have analysed the pullback just on the Verbitsky component (cfr. Remark 1.3); it is plausible that analysing what happens on the other irreducible subrepresentations one may completely determine the cohomology of M . We intend to do so in a future work.

Corollary 5.13. *We call $b_i := b_i(M)$. We have*

$$\begin{array}{c|c|c|c|c} b_{10} \leq 117877 & b_{11} \leq 30776 & b_{12} \leq 20426 & b_{13} \leq 4671 & b_{14} \leq 2623 \\ \hline b_{15} \leq 277 & b_{16} = 277 & b_{17} = 0 & b_{18} = 23 & b_{19} = 0 \end{array}$$

Proof. The estimates on b_{12} , b_{13} , b_{14} and b_{15} follow from Proposition 5.11 using the dimensions computed in table (5.1). The Betti numbers b_{16} , b_{17} , b_{18} and b_{19} are obtained from the upper bound in Proposition 5.11 and the lower bound in Corollary 5.9. \square

Corollary 5.5, Corollary 5.1, Corollary 5.9 and Corollary 5.13 prove Theorem 1.1 stated in the introduction.

6. THE COHOMOLOGY OF THE 6-DIMENSIONAL SINGULAR MODULI SPACE

Assume the notations as in Notation 2.13. In this section we discuss the case of the 6-dimensional singular O'Grady's variety $Y = K$. Using the results in Section 4 we can compute the Betti numbers of all varieties involved in the spectral sequence (3.6), that we list in the table below.

As observed at the beginning of Section 1, the variety Ω consists of 256 points and using Proposition 4.3 we get the Betti numbers of the Ω -varieties.

The variety Σ is isomorphic to $(A \times A^\vee)/\pm 1$. For the abelian 4-fold $A \times A^\vee$ we have

$$H^1(A \times A^\vee, \mathbb{Z}) = \mathbb{Z}^8 \quad H^k(A \times A^\vee, \mathbb{Z}) \simeq \Lambda^k H^1(A \times A^\vee, \mathbb{Z})$$

so that

$$b_k(A \times A^\vee) = \binom{8}{k}.$$

Finally, analysing the action of ± 1 on forms, we compute

$$H^k(\Sigma, \mathbb{Z}) = H^k(A \times A^\vee, \mathbb{Z})^{(\pm 1)^*} = \begin{cases} 0 & \text{for } k \text{ odd} \\ H^k(A \times A^\vee, \mathbb{Z}) & \text{for } k \text{ even.} \end{cases}$$

The Betti numbers of the other Σ -varieties follow from Proposition 4.8.

Finally, the cohomology of the manifold \tilde{K} has been computed in [17].

	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}
Ω	256	0	0	0	0	0	0
$\bar{\Omega}$	256	256	256	256	0	0	0
$\tilde{\Omega}$	256	256	256	256	0	0	0
$\hat{\Omega}$	256	512	512	512	256	0	0
Σ	1	28	70	28	1	0	0
$\bar{\Sigma}$	1	284	326	284	1	0	0
$\tilde{\Sigma}$	1	29	354	354	29	1	0
$\hat{\Sigma}$	1	285	610	610	285	1	0
\tilde{K}	1	8	199	1504	199	8	1

Corollary 6.1. *The Euler characteristic of K is $\chi(K) = 1208$.*

Proof. It immediately follows from the dimension computed in (6.1), as $\chi(K) = \chi(\tilde{K}) - \chi(\tilde{\Sigma}) + \chi(\Sigma)$. \square

Proposition 6.2. *We have:*

- (1) $\bar{i}^* : H^k(\bar{\Sigma}) \rightarrow H^k(\bar{\Omega})$ is surjective for any $k \geq 2$.
- (2) $\hat{i}^* : H^k(\hat{\Sigma}) \rightarrow H^k(\hat{\Omega})$ is surjective for any $k \geq 4$.
- (3) $j^* : H^2(K) \rightarrow H^2(\Sigma)$ is injective.

Proof. Since the Ω - and the Σ -varieties have no odd cohomology (cfr. (6.1)) we only need to prove the first two statements for their even cohomology groups.

(1) Following Proposition 4.8.(1) and Proposition 4.3.(2) we have:

$$H^k(\bar{\Sigma}) = q^* H^k(\Sigma) \oplus \bar{i}_* q_\Omega^* H^{k-2}(\Omega) \oplus \bar{i}_* \zeta q_\Omega^* H^{k-4}(\Omega) \oplus \bar{i}_* \zeta^2 q_\Omega^* H^{k-6}(\Omega)$$

$$H^k(\bar{\Omega}) = q_\Omega^* H^k(\Omega) \oplus \zeta q_\Omega^* H^{k-2}(\Omega) \oplus \zeta^2 q_\Omega^* H^{k-4}(\Omega) \oplus \zeta^3 q_\Omega^* H^{k-6}(\Omega)$$

For any $a \in H^l(\Omega)$ we have $\bar{i}^* \bar{i}_* q_\Omega^* a = [\bar{\Omega}]|_{\bar{\Omega}} \cdot q_\Omega^* a = 2\zeta q_\Omega^* a$, where the last equality is Proposition 4.6.(3). It follows that \bar{i}^* respects the decompositions above, and the surjectivity follows from $H^k(\Omega) = 0$ for any $k \geq 2$.

(2) Following Proposition 4.3.(3) and Proposition 4.8.(2) we have:

$$H^k(\widehat{\Sigma}) = f^* H^k(\bar{\Sigma}) \oplus c_1(\mathcal{O}_{\widehat{\Sigma}}(-\widehat{\Sigma})) \cdot f^* H^{k-2}(\bar{\Sigma})$$

$$H^k(\widehat{\Omega}) = f_\Omega^* H^k(\bar{\Omega}) \oplus h \cdot f_\Omega^* H^{k-2}(\bar{\Omega}).$$

Using the decompositions above: from the statement in part (1) we have that $\widehat{i}^*|_{f^* H^l(\bar{\Sigma})} : f^* H^l(\bar{\Sigma}) \rightarrow f_\Omega^* \bar{i}^* H^l(\bar{\Sigma}) = f_\Omega^* H^l(\bar{\Omega})$ for any $l \geq 2$, hence the claim follows from 4.6.(2).

(3) The statement is proven exactly as the injectivity of $H^2(\Sigma) \rightarrow H^2(\Omega)$ in the M case in Proposition 5.2.(1): applying [11, Theorem 2.3] we have that the restriction to Σ of the reflexive symplectic form σ_K of K is a reflexive symplectic form on Σ^1 , hence $\sigma_K \notin \ker j^*$ and we can conclude also in this case by taking a general locally trivial deformation of K .

□

Proposition 6.3. *We have*

$$H^{2k}(K) \simeq E_2^{0,2k} \quad \text{and} \quad H^{2k+1}(K) \simeq E_2^{1,2k}.$$

In particular, if non-zero the groups $H^{2k}(K)$ and $H^{2k+1}(K)$ carry a pure Hodge structure of weight $2k$.

Proof. Observe that it is enough to prove that $E_2^{2,2k} = 0$ for any $k \in \mathbb{Z}$, since all varieties appearing in the spectral sequence have trivial cohomology in odd degrees. The statement is non-trivial only for $0 \leq k \leq 8$, since otherwise $E_1^{2,k} = H^k(\widehat{\Omega}) = 0$. Since by definition $E_1^{3,k} = 0$ for any k , the statement follows once proved that the differential $d_1 : E_1^{1,k} \rightarrow E_1^{2,k}$ is surjective. Because of the description of d_1 given in (3.8), the statement follows immediately from Proposition 6.2.(2) when $k \neq 2$. For the case $k = 2$, we argue as follows. We look at the following pullback appearing in the definition of d_1 (cfr. (3.8)):

$$g_\Omega^* + f_\Omega^* : H^2(\widetilde{\Omega}) \oplus H^2(\bar{\Omega}) \rightarrow H^2(\widehat{\Omega}).$$

We show that d_1 is surjective proving that the morphism above is surjective. Because of the description of the square of Ω -varieties given in Proposition 2.12, the morphism above reads as

$$g_\Omega^* - f_\Omega^* : H^2(LG_\Omega(V)) \oplus H^2(\mathbb{P}_\Omega(V)) \rightarrow H^2(\mathbb{P}_{\bar{\Omega}}(\mathcal{U}))$$

¹The observations of Remark 5.3 apply for the varieties K and $\Sigma \subset K$ too.

and its surjectivity follows from the double decomposition of the cohomology ring of $\mathbb{P}_{\tilde{\Omega}}(\mathcal{U}) \cong \mathbb{P}_{\tilde{\Omega}}(\mathcal{L}^\perp/\mathcal{L})$, cfr. Proposition 4.2. \square

Corollary 6.4. *The pullback $\pi^*: H^{2k}(K) \rightarrow H^{2k}(\tilde{K})$ is injective for any k .*

Proof. Thanks to the above proposition, the morphism

$$(6.2) \quad \pi^* \oplus q^* i^* \oplus i_K^*: H^{2k}(K) \rightarrow H^{2k}(\tilde{K}) \oplus H^{2k}(\tilde{\Sigma}) \oplus H^{2k}(\Omega)$$

is an isomorphism on its image E_2^{2k} . By Proposition 4.9 the restriction of the projection onto the cohomology of \tilde{K}

$$(6.3) \quad E_2^{2k} \subset H^{2k}(\tilde{K}) \oplus H^{2k}(\tilde{\Sigma}) \oplus H^{2k}(\Omega) \rightarrow H^{2k}(\tilde{K})$$

is an isomorphism on its image. Taking the composition of (6.2) and (6.3) we get the claim. \square

Proposition 6.5. *We denote $e_k^{i,j} = \dim E_k^{i,j}$. We have*

$$\begin{aligned} b_{2k}(K) &\leq b_{2k}(\tilde{K}), \\ b_{2k}(K) &\geq e_1^{0,2k} - e_1^{1,2k} + e_1^{2,2k} \text{ and} \\ b_{2k+1}(K) &= e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(K). \end{aligned}$$

Proof. The first inequality follows from the very definition of W_{2k} in Proposition 4.9 and from $b_{2k}(K) = \dim W_{2k}$, see Proposition 6.3. For the second equality, consider the sequence

$$E_1^{0,2k} \xrightarrow{d_0} E_1^{1,2k} \xrightarrow{d_1} E_1^{2,2k}.$$

Using again Proposition 6.3 we have

$$b_{2k}(K) = \dim \ker d_0 = e_1^{0,2k} - \dim \operatorname{Im} d_0 \geq e_1^{0,2k} - \dim \ker d_1 = e_1^{0,2k} - e_1^{1,2k} + e_1^{2,2k}$$

where the last equality follows from the surjectivity of d_1 proven in the proof of Proposition 6.3.

For the last inequality, as $H^{2k+1}(K) \simeq \ker d_1 / \operatorname{Im} d_0$ (cfr. Proposition 6.3) we have

$$b_{2k+1}(K) = \dim \ker d_1 - \dim \operatorname{Im} d_0 = e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(K)$$

where $\dim \ker d_1 = e_1^{1,2k} - e_1^{2,2k}$ follows again from the surjectivity of d_1 . \square

Remark 6.6. Using the relation $b_{2k+1}(K) = e_1^{1,2k} - e_1^{2,2k} - e_1^{0,2k} + b_{2k}(K)$ computed in Proposition 6.5 we obtain back the Euler characteristic as computed in Corollary 6.1:

$$\begin{aligned} \chi(K) &= \sum_{k=0}^{10} (b_{2k}(K) - b_{2k+1}(K)) = \sum_{k=0}^{10} (e_1^{0,2k} - e_1^{1,2k} + e_1^{2,2k}) \\ &= \chi(\tilde{K}) + \chi(\Omega) + \chi(\tilde{\Sigma}) - \chi(\hat{\Sigma}) - \chi(\tilde{\Omega}) - \chi(\bar{\Omega}) + \chi(\hat{\Omega}) \\ &= \chi(\tilde{K}) - \chi(\tilde{\Sigma}) + \chi(\Sigma) \\ &= 1208 \end{aligned}$$

as $\chi(\tilde{\Sigma}) = \chi(\widehat{\Sigma}) - \chi(\widehat{\Omega}) + \chi(\tilde{\Omega})$ and $\chi(\Sigma) = \chi(\bar{\Sigma}) - \chi(\bar{\Omega}) + \chi(\Omega)$. In particular, the knowledge of the Euler characteristic of K does not give any further information on its Betti numbers.

Corollary 6.7. *We call $b_i := b_i(K)$. We have*

$b_0 = 1$	$b_1 = 0$	$b_2 = 23$	$b_3 = 0$	$28 \leq b_4 \leq 198$
$113 \leq b_5 \leq 283$	$1178 \leq b_6 \leq 1503$	$b_7 \leq 325$	$171 \leq b_8 \leq 199$	$b_9 \leq 28$
	$7 \leq b_{10} \leq 8$	$b_{11} \leq 1$	$b_{12} = 1$	

Proof. The estimates on the even Betti numbers are the ones in Proposition 6.5, computed via the dimensions in (6.1). The only exception is the lower bound on the 4th Betti number, that is obtained thanks to the inclusion $\text{Sym}^2 H^2(K) \hookrightarrow H^4(K)$, see [1, Proposition 5.16]. Observe that a similar inclusion holds for $2k \leq 6$, giving $\text{Sym}^k(K) \hookrightarrow H^{2k}(K)$ hence $b_{2k}(K) \geq \binom{6+k}{k}$; nevertheless, this estimate turns out to be weaker (or equal) than the one in Proposition 6.5.

For the odd Betti numbers we have used the expression in Proposition 6.5 combined with the estimates on the even Betti numbers. \square

Notice that $H^5(K, \mathbb{Q})$ carries a non trivial pure Hodge structure of weight 4, but having a Hodge structure of the “wrong” weight is not infrequent for singular varieties, as the following easy example shows.

Example 6.8. Let X be the variety obtained by gluing together 2 points p_1, p_2 of \mathbb{P}^2 and let p be the image of p_1 and p_2 in X . Using the weight spectral sequence [10, Proposition 3.3] associated to the hypercubical resolution of X

$$\begin{array}{ccc} \mathbb{P}^2 & \longleftarrow & \{p_1, p_2\} \\ \downarrow & & \downarrow \\ X & \longleftarrow & \{p\} \end{array}$$

one readily computes that $H^1(X, \mathbb{Q}) \simeq \mathbb{Q}$ with a pure Hodge structure of weight 0.

Remark 6.9. The 1st and 2nd Betti numbers were already known [21, Theorem 1.7] and [2, Lemma 2.1].

We conclude improving the estimates above with ad-hoc arguments for some of the cohomology groups of K .

Proposition 6.10. *We call $b_i := b_i(K)$. We have*

$28 \leq b_4 \leq 191$	$113 \leq b_5 \leq 276$	$1178 \leq b_6 \leq 1502$
$b_7 \leq 324$	$b_{10} = 7$	$b_{11} = 0$

Proof. Regarding the 4th and 6th Betti numbers, following Proposition 4.9 and Proposition 6.3 we need to estimate the dimension of

$$W_{2m} = \{k \in H^{2m}(\tilde{K}) : \tilde{j}^* k \in p^* H^{2m}(\Sigma)\}$$

for $m = 2, 3$. By [25, Theorem 1.5] we have an inclusion $\mathrm{Sym}^m H^2(\tilde{K}) \hookrightarrow H^{2m}(\tilde{K})$, hence using the decomposition $H^2(\tilde{K}) = \pi^* H^2(K) \oplus c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma})) \cdot \mathbb{Q}$ we obtain

$$\bigoplus_{i=0}^m c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma}))^i \cdot \mathrm{Sym}^{m-i} \pi^* H^2(K) \hookrightarrow H^{2m}(\tilde{K})$$

where we are assuming as convention $\mathrm{Sym}^0 \pi^* H^2(K) = \mathbb{Q}$. The pullback $\tilde{i}_K^* c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma}))^m$ is non zero in $H^{2m}(\tilde{\Omega})$ by Proposition 4.6.(4), hence it is not the pullback of a class in $H^{2m}(\Omega)$ via p_Ω (cfr. the decomposition Proposition 4.3.(1)) because Ω is 0-dimensional. We conclude that $c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma})) \notin W_{2m}$, which combined with Corollary 6.7 gives the upper bound on the 6th Betti number and $b_4(K) \leq 198$.

For the 4th Betti number we can give a further estimate as follows. We prove that $\tilde{j}^*(c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma})) \cdot k) \notin p^* H^4(\Sigma)$ for any $k \in \pi^* H^2(K)$, hence $c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma})) \pi^* H^2(K) \cap W_4 = (0)$ and $b_4(K) \leq 198 - b_2(K) = 191$. Note that in order to do so it is enough to prove that $\tilde{j}^* c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma})) \notin p^* H^2(\Sigma)$ since $j^* : H^2(K) \rightarrow H^2(\Sigma)$ is injective, cfr. Proposition 6.2.(3). We have

$$g^* \tilde{j}^* \mathcal{O}_{\tilde{K}}(\tilde{\Sigma}) = \hat{j}^* \gamma^* \mathcal{O}_{\tilde{K}}(\tilde{\Sigma}) = \hat{j}^* \mathcal{O}_{\hat{K}}(\hat{\Sigma} + 2\hat{\Omega}_{OG}) = \mathcal{O}_{\hat{\Sigma}}(\hat{\Sigma} + 2\hat{\Omega})$$

where the second equality was shown in Remark 4.7 and the last one holds because by definition $\hat{\Omega} = \hat{\Sigma} \cap \hat{\Omega}_{OG}$. It follows that in the decomposition given in Proposition 4.8.(2) we have

$$H^2(\hat{\Sigma}) = \underbrace{f^* H^2(\bar{\Sigma})}_{\ni c_1(\mathcal{O}_{\bar{\Sigma}}(2\hat{\Omega}))} \oplus \underbrace{c_1(\mathcal{O}_{\bar{\Sigma}}(-\hat{\Sigma})) \cdot f^* H^0(\bar{\Sigma})}_{\ni c_1(\mathcal{O}_{\bar{\Sigma}}(\hat{\Sigma}))}$$

then the class $c_1(g^* \tilde{j}^* \mathcal{O}_{\tilde{K}}(\tilde{\Sigma}))$ does not belong to $f^* H^k(\bar{\Sigma})$ hence it does not belong to $f^* q^* H^2(\Sigma) \subseteq H^2(\hat{\Sigma})$. We conclude that $\tilde{j}^* c_1(\mathcal{O}_{\tilde{K}}(\tilde{\Sigma})) \notin p^* H^2(\Sigma)$ as desired.

For the 10th Betti number, looking at the dimensions in (6.1) the differential $d_0 : E_1^{0,10} \rightarrow E_1^{1,10}$ is only given by the composite morphism

$$H^{10}(\tilde{K}) \xrightarrow{\tilde{j}^*} \underbrace{H^{10}(\tilde{\Sigma})}_{\cong \mathbb{Q}} \xrightarrow{g^*} \underbrace{H^{10}(\hat{\Sigma})}_{\cong \mathbb{Q}}$$

which is clearly surjective, hence $\dim \ker d_0 = e_1^{0,10} - e_1^{1,10} = 7$ and the claim follows from Proposition 6.3.

The estimates on the odd Betti numbers are derived from the ones on the even Betti numbers via the relation on $b_{2k+1}(K)$ in Proposition 6.5. \square

Corollary 6.4, Corollary 6.1, Corollary 6.7 and Proposition 6.10 prove Theorem 1.2 stated in the introduction.

APPENDIX A. THE STRUCTURE OF $\hat{\Omega}$ OVER $\bar{\Omega}$

Let $Y = M$. Recall from Proposition 2.11 that $\bar{\Omega} \simeq \mathbb{P}_\Omega(T\Omega) \rightarrow \Omega$ and that \mathcal{L} denotes the tautological line subbundle on $\mathbb{P}_\Omega(T\Omega)$. The purpose of this section is to prove the following proposition.

Proposition A.1. *We have an isomorphism over $\overline{\Omega}$*

$$\widehat{\Omega} \simeq \mathbb{P}_{\overline{\Omega}}(\mathcal{L}^\perp/\mathcal{L}).$$

We are going to prove it in several steps.

Recall that for a closed subscheme W of a scheme Z we can consider the normal cone to W in Z and the projective normal one:

$$C_W Z := \underline{\text{Spec}}\left(\bigoplus_{d \geq 0} I^d/I^{d+1}\right), \quad \mathbb{P}(C_W Z) := \underline{\text{Proj}}\left(\bigoplus_{d \geq 0} I^d/I^{d+1}\right)$$

where I is the ideal sheaf of W in Z .

Remark A.2. Notice that the normal cone is preserved under étale maps: given an étale map $\phi: \widetilde{Z} \rightarrow Z$ if we let \widetilde{W} be the preimage of W , then for any $w \in \widetilde{W}$

$$(C_{\widetilde{W}} \widetilde{Z})_w = (C_W Z)_{\phi(w)}.$$

Recall that $\widehat{\Omega}$ is the preimage of $\overline{\Omega}$ under the blow-up morphism $\widehat{M} = \text{Bl}_{\overline{\Sigma}} \overline{M} \rightarrow \overline{M}$. By definition of blow-up, Proposition A.1 is then equivalent to the following.

Proposition A.3. *We have an isomorphism over $\overline{\Omega}$*

$$\mathbb{P}_{\overline{\Sigma}}(C_{\overline{\Sigma}} \overline{M})|_{\overline{\Omega}} \simeq \mathbb{P}_{\overline{\Omega}}(L^\perp/L).$$

We now recall some of O'Grady's work. Recall that the variety M is the moduli space of sheaves on a K3 surface X with fixed Mukai vector $(2, 0, -2)$. As any sheaf F parametrised by M can be obtained as a quotient

$$\mathcal{O}_X(k)^{\oplus N} \rightarrow F$$

for some N and k which are independent of F , the variety M is constructed as the GIT quotient by $G := \text{PGL}(N)$ of the closure Q of the semistable locus Q^{ss} in the Quot-scheme parametrising all the quotients of $\mathcal{O}_X(k)^{\oplus N}$:

$$M = Q//G := Q^{ss}/G.$$

O'Grady in [18, §1.1] introduced a stratification of the strictly semistable locus of Q ; we will be interested in the two closed subvarieties

$$\begin{aligned} \Omega_Q &:= \overline{\{x \in Q \mid F_x \simeq I_Z \oplus I_Z, [Z] \in X^{[2]}\}}, \\ \Sigma_Q &:= \overline{\{x \in Q \mid F_x \simeq I_Z \oplus I_W, [Z], [W] \in X^{[2]}\}}. \end{aligned}$$

Notice that $\Omega_Q \subset \Sigma_Q$. Moreover, we have

$$\Omega_Q//G = \Omega \quad \text{and} \quad \Sigma_Q//G = \Sigma.$$

We let $\pi_R: R \rightarrow Q$ be the blow-up in Ω_Q , $\Sigma_R \subset R$ be the strict transform of Σ_Q and $\pi_S: S \rightarrow R$ be the blow-up in Σ_R . Thanks to Kirwan's theory of desingularisation [18, Theorem (1.2.2)], the action of G lifts to linearized actions on R and S . Using Luna's étale slice theorem, O'Grady described the normal cone to Σ_R in R .

Proposition A.4 ([18, (1.7.6), (1.7.12)]). *Let $Z \subset X$ be a subscheme of length 2 and let I_Z be its ideal sheaf. Let $x \in \Omega_Q$ be a point and F_x be the corresponding sheaf isomorphic to $I_Z \oplus I_Z = I_Z \otimes V$ with $V \simeq \mathbb{C}^2$. Let $W := \mathfrak{sl}(V)$, then*

$$\pi_R^{-1}(x) \cap \Sigma_R^{ss} \simeq \mathbb{P}\{\varphi \in \text{Hom}(W, \text{Ext}^1(I_Z, I_Z)) : \text{rk}(\varphi) \leq 1, \varphi \text{ semistable}\}.$$

Moreover, let $[\varphi] \in \pi_R^{-1}(x) \cap \Sigma_R^{ss}$ for some $\varphi \in \text{Hom}(W, \text{Ext}^1(I_Z, I_Z))$, let $St([\varphi])$ be the stabilizer and ω_φ be the symplectic form induced on $\text{Im } \varphi^\perp / \text{Im } \varphi$ by the symplectic form on $\text{Ext}^1(I_Z, I_Z)$. The normal cone of Σ_R to R at the point $[\varphi]$ is isomorphic to the normal cone of $\Sigma_R \cap \Omega_R$ to Ω_R and there exists a $St([\varphi])$ -equivariant isomorphism

$$(C_{\Sigma_R \cap \Omega_R} \Omega_R)_{[\varphi]} \xrightarrow{\sim} \{y \in \text{Hom}(\ker \varphi, \text{Im } \varphi^\perp / \text{Im } \varphi) : y^* \omega_\varphi = 0\}.$$

Further, $St([\varphi]) = \text{O}(\ker \varphi)$.

Since we want to use O'Grady's result, we first relate the variety M, \overline{M} and \widehat{M} , with Q, R and S . The blow-up morphisms $S \rightarrow R \rightarrow Q$ descend to the quotients:

$$S//G \rightarrow R//G \rightarrow Q//G.$$

Proposition A.5. *We have canonical isomorphisms*

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\sim} & S//G \\ \downarrow & & \downarrow \\ \overline{M} & \xrightarrow{\sim} & R//G \\ \downarrow & & \downarrow \\ M & \xlongequal{\quad} & Q//G. \end{array}$$

Moreover, under these identifications we have

$$\overline{\Omega}_{OG} = \Omega_R//G, \quad \overline{\Sigma} = \Sigma_R//G \quad \text{and} \quad \overline{\Omega} = \Sigma_R \cap \Omega_R//G$$

where Ω_R denotes the exceptional divisor of $R \rightarrow Q$.

Proof. Consider the composition $R^{ss} \rightarrow Q^{ss} \rightarrow M$, under which the preimage of Ω is equal to Ω_R which is a Cartier divisor. Thus, by the universal property of the blow-up we get a morphism $R^{ss} \rightarrow \overline{M}$. As it is G -invariant, it factors through the quotient $R//G$, which is an isomorphism on the complement of the exceptional divisor Ω_{OG} and maps $\Omega_R//G$ onto $\overline{\Omega}_{OG}$.

We want to show that the surjection $\Omega_R//G \rightarrow \overline{\Omega}_{OG}$ is an isomorphism, in other words we are left to prove the following

Claim. Let I be the ideal of Ω_Q in Q and J be the ideal of Ω in M . The morphism

$$\underline{\text{Proj}}\left(\bigoplus_{d \geq 0} I^d / I^{d+1}\right) // G = \Omega_R // G \rightarrow \overline{\Omega}_{OG} = \underline{\text{Proj}}\left(\bigoplus_{d \geq 0} J^d / J^{d+1}\right).$$

is an isomorphism.

Using Luna's étale slice theorem [18, Theorem (1.2.1)] we first reduce to deal with an affine quotient. Indeed, for any $\omega \in \Omega_R$ we can find a $St(\omega)$ -stable affine open set $\omega \in \mathcal{V} \subset Q$, such that the multiplication morphism

$$G \times_{St(\omega)} \mathcal{V} \xrightarrow{\phi} Q$$

has open image, is étale over its image and is G -equivariant (here $St(\omega)$ acts on $G \times_{St(\omega)} \mathcal{V}$ by $h(g, p) := (gh^{-1}, hp)$ for any $p \in \mathcal{V}$). Let $\pi: \mathcal{V} \rightarrow \bar{\mathcal{V}} := \mathcal{V} // St(\omega)$ be the quotient and $\bar{\omega}$ be the image of ω . The quotient map

$$\bar{\phi}: \bar{\mathcal{V}} \rightarrow Q // G = \bar{M}$$

has open image and is étale over its image. For such a \mathcal{V} we have that [18, (1.2.2)]

$$(C_{\mathcal{V} \cap \Omega_Q} \mathcal{V})_{\omega} \simeq (C_{\Omega_Q} Q)_{\omega}.$$

As $\bar{\phi}$ is étale we are left to prove

$$\mathbb{P}(C_{\mathcal{V} \cap \Omega_Q} \mathcal{V})_{\omega} // St(\omega) \simeq \mathbb{P}(C_{\bar{\mathcal{V}} \cap \pi(\Omega_Q)} \bar{\mathcal{V}})_{\bar{\omega}}.$$

Let I be the ideal of $\mathcal{V} \cap \Omega_Q$ in \mathcal{V} . As $St(\omega)$ is reductive [18, Corollary (1.1.8)], then $I^{St(\omega)}$ equals the ideal of $\mathcal{V} // St(\omega) \cap \pi(\Omega_Q)$ in $\mathcal{V} // St(\omega)$. Thus we have proven at once that $\Omega_R // G \simeq \bar{\Omega}_{OG}$ and that $\bar{M} \simeq R // G$.

By construction we have an isomorphism $(\Sigma_R \setminus \Omega_R) // G \xrightarrow{\sim} \bar{\Sigma} \setminus \bar{\Omega}$ and we conclude $\Sigma_R // G = \bar{\Sigma}$ by irreducibility. Thanks to this last equality one shows with an analogous reasoning $S // G \simeq \widehat{M}$.

Finally,

$$(\Sigma_R \cap \Omega_R) // G = \Sigma_R // G \cap \Omega_R // G = \bar{\Sigma} \cap \bar{\Omega}_{OG} = \bar{\Omega},$$

where the first equality holds because R^{ss}/G is a good quotient. □

Lemma A.6. *Let $x \in \Omega_Q$, let $[\varphi]$ be a point in $\pi_R^{-1}(x) \cap \Sigma_R^{ss}$ and $\bar{\varphi}$ its image under the quotient map $\Omega_R \cap \Sigma_R^{ss} \rightarrow \bar{\Omega}$. Then*

$$\mathbb{P}(C_{\bar{\Sigma} \bar{M}})_{\bar{\varphi}} \simeq \mathbb{P}(C_{\Sigma_R R})_{[\varphi]} // St([\varphi]).$$

Proof. Using the results in Proposition A.5 and that $St([\varphi]) = O(\ker \varphi)$ is also reductive, the proof is completely similar to the proof above. □

Proof of Proposition A.1. Notice that the above mentioned description of the cone due to O'Grady works in an analytic neighbourhood of any point $[\varphi] \in \Omega_R \cap \Sigma_R^{ss}$ and that in such a neighbourhood we have the natural isomorphism

$$\mathbb{P}(C_{\Sigma R})_{[\varphi]} // St([\varphi]) \rightarrow \mathbb{P}(\mathcal{L}^{\perp} / \mathcal{L})_{\bar{\varphi}}, [y] \mapsto [\text{Im } y].$$

Combining this with Lemma A.6 we get the desired isomorphism

$$\mathbb{P}(C_{\bar{\Sigma} \bar{M}})_{\bar{\Omega}} \rightarrow \mathbb{P}(\mathcal{L}^{\perp} / \mathcal{L}).$$

□

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