Dynamic boundary conditions for divergence form operators with Hölder coefficients

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Abstract

We consider a second-order elliptic operator in divergence form with merely Hölder continuous coefficients on a bounded domain Ω with $C^{1,\kappa}$-boundary Γ with Wentzell boundary conditions of the type $\operatorname{Tr} Au = \beta\, \partial_\nu u + \alpha \operatorname{Tr} u$ on Γ. For strictly positive bounded measurable β we prove maximal regularity on $L_p(\Omega) \times L_p(\Gamma)$ for all $p \in (1,\infty)$, the generation of a holomorphic C_0-semigroup with angle $\frac{\pi}{2}$ for all $p \in [1,\infty)$ and also the generation of a holomorphic C_0-semigroup with angle $\frac{\pi}{2}$ on $C(\overline{\Omega})$.

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1 Introduction

Recently there has been a lot of interest in parabolic systems with dynamic (Wentzell) boundary conditions

$$\left[\begin{array}{ll} \dfrac{d}{dt}\, u(t,\cdot) = -B_m u(t,\cdot) & \text{on } \Omega, \\[2ex] \dfrac{d}{dt} \operatorname{Tr} u(t,\cdot) = -\beta\, \partial_\nu u(t,\cdot) - \alpha \operatorname{Tr} u(t,\cdot) & \text{on } \partial\Omega, \\[2ex] u(0,\cdot) = u_0 & \text{on } \Omega. \end{array}\right. \tag{1}$$

Here $\Omega \subset \mathbb{R}^d$ is an open, bounded and connected set, B_m is a second-order elliptic operator, $\alpha \in L_\infty(\Omega)$, the function $\beta \colon \Omega \to (0,\infty)$ is bounded measurable with $\operatorname{ess\,inf} \beta > 0$ and ∂_ν is the outward co-normal derivative associated with the operator B_m. The system (1) can be rewritten on the product space $\Omega \oplus \partial\Omega$ in matrix form by

$$\frac{d}{dt} \begin{pmatrix} u(t,\cdot) \\ \varphi(t,\cdot) \end{pmatrix} = -\mathbb{A} \begin{pmatrix} u(t,\cdot) \\ \varphi(t,\cdot) \end{pmatrix}, \quad \begin{pmatrix} u(0,\cdot) \\ \varphi(0,\cdot) \end{pmatrix} = \begin{pmatrix} u_0 \\ \operatorname{Tr} u_0 \end{pmatrix}, \tag{2}$$

where

$$\mathbb{A} = \begin{pmatrix} B_m & 0 \\ \beta\,\partial_\nu & \alpha \end{pmatrix}$$

and formally $D(\mathbb{A}) \subset \{(v,\varphi) : \operatorname{Tr} v = \varphi\}$. Typical questions are whether $\mathbb{A}$ generates a C_0-semigroup, whether this semigroup is holomorphic and if so, what is the holomorphy angle. Another question is whether the operator $\mathbb{A}$ has maximal L_r-regularity for all $r \in (1,\infty)$.

Operators with Wentzell boundary conditions have been first studied by Wentzell [Ven] and Feller [Fel]. Hintermann [Hin] studied elliptic operators with dynamic or Wentzell boundary conditions on C^∞-domains and proved generation of strongly continuous semigroups. Amann–Escher [AmE] considered C^2-domains and operators in divergence form with uniformly continuous symmetric principal coefficients and $\beta = \mathbb{1}_\Gamma$, and proved on $C(\overline{\Omega})$ and for all $p \in [1,\infty)$ on $L_p(\Omega) \times L_p(\partial\Omega)$ the generation of a positive contraction semigroup. Since the operator is self-adjoint on L_2 they obtained by interpolation that the semigroup is holomorphic on L_p for all $p \in (1,\infty)$. In [FGGR] Favini et. al. studied degenerate operators of the form $\operatorname{div}(a\nabla\cdot)$ with Wentzell boundary conditions on C^2-domains with $a \in C^1(\overline{\Omega})$ and $\beta \in C^1(\partial\Omega)$, and proved similar results. The holomorphy on L_1 has been proved by Warma [War] for the Laplacian on C^∞-domains with $\beta \in C^1(\partial\Omega)$ and he also proved that the holomorphy angle is equal to $\frac{\pi}{2}$. In [FGG$^+$2] Favini et. al. extended these results on L_p for all $p \in [1,\infty)$ to arbitrary uniformly elliptic operators in divergence form on C^∞-domains with C^∞ principal coefficients and $\beta \in C^\infty(\Gamma)$ without proving the optimal angle of holomorphy.

Assuming Ω merely Lipschitz and β measurable, Arendt et. al. showed in [AMPR] that the Laplacian with Wentzell boundary conditions generates a strongly continuous semigroup on L_p for all $p \in [1,\infty)$ with holomorphy if $p \in (1,\infty)$. On $C(\overline{\Omega})$ they proved generation of a C_0-semigroup if Ω is of class $C^{2,\kappa}$ with $\kappa > 0$ and β continuous. Moreover, they showed that the semigroup is ultracontractive. Engel [Eng] proved that the Laplacian with Wentzell boundary conditions on $C(\overline{\Omega})$ generates a holomorphic C_0-semigroup with angle $\frac{\pi}{2}$ if Ω is of class C^∞. Engel–Fragnelli [EF] extended this result to arbitrary uniformly elliptic operators in divergence form with C^∞ principal coefficients on C^∞-domains and $\beta = \mathbb{1}_\Gamma$ without proving the optimal angle of holomorphy. In [BE2] the authors generalized and proved that uniformly elliptic operators in divergence form with Lipschitz continuous principal coefficients on $C^{1,1}$-domains and $\beta = \mathbb{1}_\Gamma$ generate holomorphic semigroups of optimal angle $\frac{\pi}{2}$. Moreover in [Bin2], the same results were proved on smooth, compact, Riemannian manifolds with smooth boundary.

In [DPZ] Denk, Prüss and Zacher discussed the question of maximal L_r-regularity for uniformly elliptic operators in non-divergence form with continuous principal coefficients on $L_p(\Omega) \times L_p(\partial\Omega)$ on C^∞-domains with dynamic boundary conditions, with $p, r \in (1,\infty)$. Recently, in [GGGR] Goldstein et. al. proved maximal L_r-regularity for uniformly elliptic operators in non-divergence form with continuous principal coefficients on $L_p(\Omega) \times L_p(\partial\Omega)$ on C^2-domains with generalized Wentzell boundary conditions, with $p, r \in (1,\infty)$. Their boundary conditions are dynamic boundary conditions but with an additional elliptic second-order operator on the boundary. For such boundary conditions see also [FGG$^+$2] and [Bin1].

There are two ways to define a co-normal derivative of a function u. One (classical) way is to define it as $\sum_{k,l=1}^d \nu_k \, c_{kl} \, \partial_l u$, where $(\nu_1, \ldots, \nu_d)$ is the outward normal derivative and the c_{kl} are the principal coefficients of B_m. The second way is to define it as a weak co-normal derivative in $L_2(\Gamma)$ via a Gauss–Green formula (see (5)). In the smooth setting one can apply the divergence theorem to the vector valued function $F \colon \Omega \to \mathbb{C}^d$ given by $F_k = \sum_{l=1}^d c_{kl} \, \partial_l u$ to obtain that the two notions of co-normal derivative coincide. A sufficient condition to apply the divergence theorem is that the $c_{kl} \in W^{1,\infty}(\Omega)$. The above mentioned proofs for the holomorphy on $C(\overline{\Omega})$ use the classical notion of normal derivative. For the main results in this paper we assume that Ω is of class $C^{1,\kappa}$, with $\kappa \in (0,1)$ and B_m is a second-order operator in divergence form with real uniform Hölder continuous coefficients plus a real valued bounded measurable potential. Since the principal coefficients are in general not Lipschitz continuous, the divergence theorem is not applicable. This gives a major complication. For the functions α and β we require that $\beta \colon \Omega \to (0, \infty)$ is a bounded measurable function with $\operatorname{ess\,inf} \beta > 0$ and $\alpha \in L_\infty(\Omega)$. In this setting we shall prove in Section 2 via form methods as in [AmE], [AMPR] and [AE2] that $-\mathbb{A}$ generates a C_0-semigroup S on $L_2(\Omega) \times L_2(\partial\Omega)$ which is holomorphic with (semi-)angle $\frac{\pi}{2}$. This construction involves the weak co-normal derivative. Moreover, we shall prove in Theorem 4.1 and Corollary 5.6 that S extends consistently to a C_0-semigroup on $L_p(\Omega) \times L_p(\partial\Omega)$ for all $p \in [1, \infty)$ and that the semigroup is holomorphic with optimal angle $\frac{\pi}{2}$. In addition we prove that the generator on $L_p(\Omega) \times L_p(\partial\Omega)$ has maximal L_r-regularity for all $p, r \in (1, \infty)$. We also prove that the part of $\mathbb{A}$ in $\{(u, u|_\Gamma) : u \in C(\overline{\Omega})\}$ generates a C_0-semigroup which is holomorphic with angle $\frac{\pi}{2}$. We emphasise that also on $C(\overline{\Omega})$ we do not require that β is continuous.

As in [CENN], [Eng], [EF] and [BE1] we use a similarity transformation to write the transformed image of the operator $\mathbb{A}$ as $\begin{pmatrix} B^D & 0 \\ 0 & \beta \mathcal{N} \end{pmatrix}$ plus a perturbation, where B^D is the elliptic operator with Dirichlet boundary conditions and $\mathcal{N}$ is the Dirichlet-to-Neumann operator, under the condition that the operator B^D is invertible. We shall show in Corollary 3.9 that $-\beta \mathcal{N}$ generates a C_0-semigroup on $L_2(\partial\Omega)$ which extends consistently to a C_0-semigroup on $L_p(\partial\Omega)$ and the latter semigroup is holomorphic with angle $\frac{\pi}{2}$ for all $p \in [1, \infty)$. Moreover, the semigroup extends to a holomorphic C_0-semigroup on $C(\partial\Omega)$ with angle $\frac{\pi}{2}$, see Corollary 3.12. In order to prove this we first show that the semigroup generated by $-\beta \mathcal{N}$ has Poisson kernel bounds on the right half-plane, using techniques developed in [EO1], [EO2] and [EO3]. Then Hieber–Prüss [HP] implies that $\beta \mathcal{N}$ has maximal L_r-regularity on $L_p(\partial\Omega)$ for all $p, r \in (1, \infty)$. A perturbation result of Kunstmann–Weis [KW] then gives maximal regularity of the operator $\mathbb{A}$ in $L_p(\Omega) \times L_p(\partial\Omega)$ for all $p \in (1, \infty)$.

2 The operator in L_2

In this section we introduce almost every notation that we need in this paper and construct the operator on L_2.

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and connected set with Lipschitz boundary. Write $\Gamma = \partial\Omega$. We provide Γ with the $(d-1)$-dimensional Hausdorff measure, denoted by σ.

For all $k, l \in \{1, \ldots, d\}$ let $c_{kl}, c_0 \colon \Omega \to \mathbb{R}$ be bounded measurable functions with $c_{kl} = c_{lk}$. We write $C(x) = (c_{kl}(x))_{k,l\in\{1,\ldots,d\}} \in M^{d\times d}(\mathbb{R})$ for all $x \in \Omega$. Further, let $\alpha \colon \Gamma \to \mathbb{C}$ be a bounded measurable function and let $\beta \colon \Gamma \to (0, \infty)$ be a bounded measurable function such that $\operatorname{ess\,inf} \beta > 0$. We assume that there exists a $\mu > 0$ such that $\sum_{k,l=1}^d c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Note that $\frac{1}{\beta}$ is a bounded function. Define the form $\mathfrak{a} \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (\partial_k u)\, \overline{\partial_l v} + \int_\Omega c_0\, u\, \overline{v} + \int_\Gamma \frac{\alpha}{\beta}\, (\operatorname{Tr} u)\, \overline{\operatorname{Tr} v}\, \mathrm{d}\sigma.$$

It is well known that $\mathfrak{a}$ is a continuous elliptic form. Define

$$\mathbb{L}_2 := L_2(\Omega) \times L_2(\Gamma)$$

equipped with the norm

$$\|(u, \varphi)\|_{\mathbb{L}_2}^2 = \int_\Omega |u|^2 + \int_\Gamma |\varphi|^2\, \frac{d\sigma}{\beta},$$

where we recall that σ is the $(d-1)$-dimensional Hausdorff measure on Γ. Note the factor β in the norm.

Define $j \colon W^{1,2}(\Omega) \to \mathbb{L}_2$ by

$$j(u) = (u, \operatorname{Tr} u).$$

Then j is continuous and has dense range. Moreover, for all $\theta \in (0, \frac{\pi}{2})$ there exists an $\omega > 0$ such that

$$\mathfrak{a}(u, u) + \omega\, \|j(u)\|_{\mathbb{L}_2}^2 \in \Sigma_\theta,$$

where $\Sigma_\theta = \{re^{i\eta} : r \in [0, \infty) \text{ and } \eta \in [-\theta, \theta]\}$. Note that $\mathfrak{a}(u, u)$ is not a real number in general, since the function α is complex valued. We define the **variational operator** $\mathbb{A}$ to be the m-sectorial operator in $\mathbb{L}_2$ associated with $(\mathfrak{a}, j)$, see [AE2]. Then $-\mathbb{A}$ is the generator of a C_0-semigroup which is holomorphic in the right half-plane. By definition for all $(u, \varphi), (f, \eta) \in \mathbb{L}_2$ one has that $(u, \varphi) \in D(\mathbb{A})$ and $\mathbb{A}(u, \varphi) = (f, \eta)$ if and only if

$$\left[\begin{array}{l} u \in W^{1,2}(\Omega), \\ \varphi = \operatorname{Tr} u, \text{ and} \\ \mathfrak{a}(u, v) = \displaystyle\int_\Omega f\, \overline{v} + \int_\Gamma \eta\, \overline{\operatorname{Tr} v}\, \frac{\mathrm{d}\sigma}{\beta} \quad \text{for all } v \in W^{1,2}(\Omega). \end{array}\right. \tag{3}$$

In order to characterise the generator we introduce some more notation. Define the form $\mathfrak{b}\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{b}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (\partial_k u)\, \overline{\partial_l v} + \int_\Omega c_0\, u\, \overline{v}. \tag{4}$$

Further define $\mathcal{B}\colon W^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle \mathcal{B}u, \tau \rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \mathfrak{b}(u,\tau).$$

We need the notion of a weak co-normal derivative. If $u \in W^{1,2}(\Omega)$ and $\psi \in L_2(\Gamma)$, then we say that $u \in D(\partial_\nu^C)$ and $\partial_\nu^C u = \psi$ if $\mathcal{B}u \in L_2(\Omega)$ and

$$\mathfrak{b}(u,v) - \int_\Omega (\mathcal{B}u)\, \overline{v} = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}\, \mathrm{d}\sigma \tag{5}$$

for all $v \in W^{1,2}(\Omega)$. It follows from the Stone–Weierstraß theorem that the function ψ is indeed unique. We say that ψ is the **(weak) co-normal derivative** of u. Note that the co-normal derivative is independent of c_0 and that $D(\partial_\nu^C) \subset W^{1,2}(\Omega)$. Obviously the co-normal derivative depends on the matrix valued function C of principal coefficients. This is why we denote it by $\partial_\nu^C u$.

Lemma 2.1. *Let $(u,\varphi), (f,\eta) \in \mathbb{L}_2$. Then the following are equivalent.*

(i) $(u,\varphi) \in D(\mathbb{A})$ *and* $\mathbb{A}(u,\varphi) = (f,\eta)$.

(ii) $u \in D(\partial_\nu^C)$, $\varphi = \mathrm{Tr}\, u$, $f = \mathcal{B}u$ *and* $\eta = \beta\, \partial_\nu^C u + \alpha\, \mathrm{Tr}\, u$.

Proof. The proof is similar to the proof of [AE2] Proposition 4.17.

'(i)⇒(ii)'. Choosing $v \in C_c^\infty(\Omega)$ it follows from (3) that $\mathcal{B}u = f$. Then

$$\int_\Omega (\mathcal{B}u)\, \overline{v} + \int_\Gamma \eta\, \overline{\mathrm{Tr}\, v}\, \frac{\mathrm{d}\sigma}{\beta} = \mathfrak{a}(u,v) = \mathfrak{b}(u,v) + \int_\Gamma \frac{\alpha}{\beta}\, (\mathrm{Tr}\, u)\, \overline{\mathrm{Tr}\, v}\, \mathrm{d}\sigma$$

for all $v \in W^{1,2}(\Omega)$. Therefore

$$\int_\Gamma \eta\, \overline{\mathrm{Tr}\, v}\, \frac{\mathrm{d}\sigma}{\beta} - \int_\Gamma \frac{\alpha}{\beta} (\mathrm{Tr}\, u)\, \overline{\mathrm{Tr}\, v}\, \mathrm{d}\sigma = \mathfrak{b}(u,v) - \int_\Omega (\mathcal{B}u)\, \overline{v}$$

So u has a co-normal derivative and $\partial_\nu^C u = \frac{\eta}{\beta} - \frac{\alpha}{\beta}\, \mathrm{Tr}\, u$.

'(ii)⇒(i)'. The proof is similar. □

So $D(\mathbb{A}) = \{(u, \mathrm{Tr}\, u) : u \in D(\partial_\nu^C)\}$. Lemma 2.1 gives a precise meaning that $(t,x) \mapsto (e^{-t\mathbb{A}}(u_0, \mathrm{Tr}\, u_0))(x)$ satisfies (1) and (2) in the introduction.

The next perturbation result allows to restrict to the case that $c_0 \geq 0$. It follows immediately from Lemma 2.1.

Lemma 2.2. *Let $\lambda \in \mathbb{R}$ and let $\mathbb{A}^\sharp$ be the operator similar to $\mathbb{A}$, but with c_0 replaced by $c_0 + \lambda \mathbb{1}_\Omega$. Then $D(\mathbb{A}^\sharp) = D(\mathbb{A})$ and*

$$\mathbb{A}^\sharp(u,\varphi) = \mathbb{A}(u,\varphi) + (\lambda\, u, 0)$$

for all $(u,\varphi) \in D(\mathbb{A})$.

We next describe via a similarity transformation the operator $\mathbb{A}$ as an operator in $W^{1,2}(\Omega)$ with Wentzell boundary conditions. This was done in [FGG+1] Theorem 2.1 for the Laplacian and we adapt the argument given in [AE2] Proposition 4.19.

Proposition 2.3. *Define the operator A in the Hilbert space $W^{1,2}(\Omega)$ by*

$$D(A) = \{u \in D(\partial_\nu^C) : \mathcal{B}u \in W^{1,2}(\Omega) \text{ and } \beta\, \partial_\nu^C u = \operatorname{Tr} \mathcal{B}u - \alpha \operatorname{Tr} u\}$$

and $Au = \mathcal{B}u$. Then $-A$ generates a holomorphic C_0-semigroup on $W^{1,2}(\Omega)$.

Proof. Since j is injective, one can transfer the form $\mathfrak{a}$ on $W^{1,2}(\Omega)$ to a form $\widetilde{\mathfrak{a}}$ on $j(W^{1,2}(\Omega))$ by defining $D(\widetilde{\mathfrak{a}}) = j(W^{1,2}(\Omega))$ and $\widetilde{\mathfrak{a}}(j(u), j(v)) = \mathfrak{a}(u,v)$ for all $u, v \in W^{1,2}(\Omega)$. Then $\widetilde{\mathfrak{a}}$ is a densely defined closed sectorial form in $\mathbb{L}_2$ and $\mathbb{A}$ is the operator associated with $\widetilde{\mathfrak{a}}$. We provide $D(\widetilde{\mathfrak{a}})$ with the norm $\|j(u)\|_{D(\widetilde{\mathfrak{a}})} = \|u\|_{W^{1,2}(\Omega)}$ for all $u \in W^{1,2}(\Omega)$. Let $\widetilde{A}$ be the part of $\mathbb{A}$ in $D(\widetilde{\mathfrak{a}})$. So $D(\widetilde{A}) = \{F \in D(\widetilde{\mathfrak{a}}) : \mathbb{A}F \in D(\widetilde{\mathfrak{a}})\}$. Then $-\widetilde{A}$ is the generator of a holomorphic C_0-semigroup in $D(\widetilde{\mathfrak{a}})$ by Proposition 2.4 below. Define $J \colon W^{1,2}(\Omega) \to D(\widetilde{\mathfrak{a}})$ by $J(u) = j(u)$. Then J is an isomorphism, so $-J^{-1}\,\widetilde{A}\,J$ is the generator of a holomorphic C_0-semigroup on $W^{1,2}(\Omega)$. It remains to show that $A = J^{-1}\,\widetilde{A}\,J$.

Let $u \in D(A)$. Then $\mathcal{B}u \in W^{1,2}(\Omega)$ and $\beta\,\partial_\nu^C u = \operatorname{Tr}\mathcal{B}u - \alpha \operatorname{Tr} u$. So $j(u) = (u, \operatorname{Tr} u) \in D(\mathbb{A})$ and $\mathbb{A}j(u) = (\mathcal{B}u, \operatorname{Tr}\mathcal{B}u) = j(\mathcal{B}u) \in D(\widetilde{\mathfrak{a}})$ by Lemma 2.1 (ii)⇒(i). Hence $j(u) \in D(\widetilde{A})$ and $u \in D(J^{-1}\,\widetilde{A}\,J)$. Moreover, $J^{-1}\,\widetilde{A}\,Ju = \mathcal{B}u = Au$.

Conversely, let $u \in D(J^{-1}\,\widetilde{A}\,J)$. Then $j(u) \in D(\widetilde{\mathfrak{a}})$ and $\mathbb{A}j(u) \in D(\widetilde{\mathfrak{a}})$. So $u \in W^{1,2}(\Omega)$ and by Lemma 2.1 (i)⇒(ii) one deduces that $\mathcal{B}u \in W^{1,2}(\Omega)$ and $\beta\,\partial_\nu^C u = \operatorname{Tr}\mathcal{B}u - \alpha \operatorname{Tr} u$. So $u \in D(A)$. □

In the proof we used the following general result.

Proposition 2.4. *Let V and H be Hilbert spaces with V continuously and densely embedded in H. Let $\mathfrak{t} \colon V \times V \to \mathbb{C}$ be a continuous elliptic sesquilinear form. Let T be the m-sectorial operator in H associated with $\mathfrak{t}$ and let $\widetilde{T}$ be the part of T in V. Then $-\widetilde{T}$ is the generator of a holomorphic C_0-semigroup in V.*

Proof. Without loss of generality we may assume that $\mathfrak{t}$ is coercive. Define $\mathcal{T} \colon V \to V^*$ by $(\mathcal{T}u)(v) = \mathfrak{t}(u,v)$. Then $\mathcal{T}$ is a topological isomorphism by the Lax–Milgram theorem. Consider $\mathcal{T}$ as a densely defined operator in V^*, where we use the Gelfand tripple (V, H, V^*). Then $-\mathcal{T}$ is the generator of a holomorphic C_0-semigroup in V^* by [Ouh2] Theorem 1.55. Using the topological isomorphism we obtain that $(\mathcal{T}^{-1}\, e^{-t\mathcal{T}}\, \mathcal{T})_{t>0}$ is a holomorphic C_0-semigroup in V. The generator of the latter semigroup is $-\mathcal{T}^{-1}\,\mathcal{T}^2$, which is equal to $-\widetilde{T}$. □

3 Multiplicative perturbation of the Dirichlet-to-Neumann operator

As an intermediate result, which is of independent interest, we study in this section a multiplicative perturbation of the Dirichlet-to-Neumann operator.

We adopt the notation and assumptions as in Section 2. Recall that Ω has a Lipschitz boundary in Section 2. Define the form $\mathfrak{b}_D \colon W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{C}$ by $\mathfrak{b}_D = \mathfrak{b}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Then $\mathfrak{b}_D$ is a closed sectorial form in $L_2(\Omega)$. Let B_2^D be the operator associated to $\mathfrak{b}_D$. Throughout this section we assume in addition that $0 \notin \sigma(B_2^D)$.

We frequently need the notion of consistent operators and semigroups. Let X_0 and X_1 be two Banach spaces which are embedded in a vector space X. Let Y be a vector space. Further let $T_0 \colon D(T_0) \to Y$ and $T_1 \colon D(T_1) \to Y$ be two operators with domains $D(T_0) \subset X_0$ and $D(T_1) \subset X_1$. Then the operators T_0 and T_1 are called **consistent** if $T_0 x = T_1 x$ for all $x \in D(T_0) \cap D(T_1)$. In order to avoid a domain problem for unbounded operators we call the operators T_0 and T_1 **strongly consistent** if they are consistent and in addition

$$X_1 \subset X_0 \text{ and } D(T_1) \subset D(T_0), \text{ or}$$

$$X_0 \subset X_1 \text{ and } D(T_0) \subset D(T_1).$$

If T_0 and T_1 are strongly consistent and $\lambda \in \rho(T_0) \cap \rho(T_1)$, then it is easy to verify that the resolvent $(\lambda I - T_0)^{-1}$ is consistent with $(\lambda I - T_1)^{-1}$. Let $S^{(0)}$ and $S^{(1)}$ be semigroups in X_0 and X_1. Then the semigroups $S^{(0)}$ and $S^{(1)}$ are called **consistent** if $S^{(0)}_t$ and $S^{(1)}_t$ are consistent for all $t > 0$.

The next lemma is useful to relate consistent C_0-semigroups and [strongly] consistent generators.

Lemma 3.1. *Let X_0 and X_1 be two Banach spaces which are embedded in a vector space X. Let $S^{(0)}$ and $S^{(1)}$ be C_0-semigroups in X_0 and X_1 with generators $-A_0$ and $-A_1$. Then one has the following.*

(a) *If $S^{(0)}$ and $S^{(1)}$ are consistent, then A_0 and A_1 are consistent and*

$$D(A_0) \cap D(A_1) = \{x \in D(A_1) \cap X_0 : A_1 x \in X_0\}. \tag{6}$$

(b) *If $S^{(0)}$ and $S^{(1)}$ are consistent and $X_1 \subset X_0$ (or $X_0 \subset X_1$), then A_0 and A_1 are strongly consistent.*

(c) *If A_0 and A_1 are strongly consistent, then $S^{(0)}$ and $S^{(1)}$ are consistent.*

Proof. '(a)'. See [ER] Proposition 2.5.

'(b)'. It follows from (6) that $D(A_1) \subset D(A_0)$. The rest is obvious.

'(c)'. Without loss of generalities we may assume that $S^{(0)}$ and $S^{(1)}$ are bounded semigroups. Then $(-\infty, 0) \subset \rho(A_0) \cap \rho(A_1)$ and $(\lambda I + A_0)^{-1}$ is consistent with $(\lambda I + A_1)^{-1}$ for all $\lambda \in (0, \infty)$. Then the result follows from [ER] Lemma 2.3. $\square$

Recall that σ is the $(d-1)$-dimensional Hausdorff measure on Γ and the space $L_p(\Gamma)$ is with respect to the measure σ for all $p \in [1, \infty)$. We also need the Dirichlet-to-Neumann operator $\mathcal{N}_p$ on $L_p(\Gamma)$ for all $p \in [1, \infty)$. Let $\mathcal{N}$ be the self-adjoint operator associated with $(\mathfrak{b}, \mathrm{Tr}\,)$ and let $T^{(2)}$ be the semigroup generated by $-\mathcal{N}$, see [AEKS] Theorem 4.5. Note that $\mathcal{N}$ is an operator, not a multivalued graph, since $0 \notin \sigma(B_2^D)$ (cf. [AEKS] Proposition 4.11.) If

(I) $c_0 \geq 0$ or,

(II) there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ,

then the semigroup $T^{(2)}$ extends consistently to a semigroup $T^{(p)}$ on $L_p(\Gamma)$ for all $p \in [1, \infty]$ and $T^{(p)}$ is a C_0-semigroup if $p \in [1, \infty)$. This follows from [EO2] Theorem 2.2(b) in Case (I) and from [EO2] Lemma 8.1 in Case (II). We denote by $-\mathcal{N}_p$ the generator of $T^{(p)}$. So $\mathcal{N}_2 = \mathcal{N}$. In Case (II) the semigroup $T^{(p)}$ is holomorphic with angle $\frac{\pi}{2}$ for all $p \in [1, \infty)$ by [EO3] Proposition 3.3.

Recall that $\beta \colon \Gamma \to (0, \infty)$ is a measurable function with $\operatorname{ess\,inf} \beta > 0$. We denote by M_β the multiplication operator with β on $L_p(\Gamma)$.

The following proposition is inspired by [AE2] Proposition 4.10.

Proposition 3.2. *Suppose*

(I) $c_0 \geq 0$ *or,*

(II) *there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ.*

Then one has the following.

(a) *The operator $M_\beta \, \mathcal{N} \, M_\beta$ is self-adjoint and lower bounded.*

(b) *The semigroup generated by $-M_\beta \, \mathcal{N} \, M_\beta$ extends consistently to a semigroup on $L_p(\Gamma)$ for all $p \in [1, \infty]$, which is a C_0-semigroup if $p \in [1, \infty)$ with generator $-M_\beta \, \mathcal{N}_p \, M_\beta$.*

(c) *There exist $c, \omega > 0$ such that*

$$\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{L_p(\Gamma) \to L_q(\Gamma)} \leq c \, t^{-(d-1)(\frac{1}{p} - \frac{1}{q})} \, e^{\omega t} \tag{7}$$

for all $t > 0$ and $p, q \in [1, \infty]$ with $p \leq q$.

Proof. The proof is divided in several steps. We first prove the proposition in Case (I). Near the end we prove Case (II) via a perturbation argument.

Step 1 Clearly the operator $M_\beta \, \mathcal{N} \, M_\beta$ is self-adjoint and lower bounded, which is Statement (a). We describe it with form methods. Since $c_0 \geq 0$, the form $\mathfrak{b}$ is $\frac{1}{\beta} \,\mathrm{Tr}\,$-elliptic. This follows as in Step 1 of the proof of Proposition 3.3 in [AM]. Let $\widetilde{\mathcal{N}}$ denote the operator associated with $(\mathfrak{b}, \frac{1}{\beta} \,\mathrm{Tr}\,)$. If $\varphi \in D(\widetilde{\mathcal{N}})$ and $\psi = \widetilde{\mathcal{N}} \varphi$, then there there exists a

$u \in W^{1,2}(\Omega)$ such that $\frac{1}{\beta} \operatorname{Tr} u = \varphi$ and $\mathfrak{b}(u,v) = (\psi, \frac{1}{\beta} \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. Then $\mathfrak{b}(u,v) = (\frac{1}{\beta}\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. So $\operatorname{Tr} u \in D(\mathcal{N})$ and $\mathcal{N}\operatorname{Tr} u = \frac{1}{\beta}\psi$. Hence $\widetilde{\mathcal{N}} \subset M_\beta \mathcal{N} M_\beta$. The converse inclusion can be proved similarly. So $\widetilde{\mathcal{N}} = M_\beta \mathcal{N} M_\beta$.

Step 2 Let $C = \{\varphi \in L_2(\Gamma, \mathbb{R}) : \varphi \leq \frac{1}{\beta}\}$. We shall prove that C is invariant under the semigroup generated by $-M_\beta \mathcal{N} M_\beta$. The set C is closed and convex in $L_2(\Gamma)$. Define $P \colon L_2(\Gamma) \to C$ by $P\varphi = \frac{1}{\beta} \mathbb{1}_\Gamma \wedge \operatorname{Re} \varphi$. Then P is the orthogonal projection onto C. Let $u \in W^{1,2}(\Omega)$. Define $w = \mathbb{1}_\Omega \wedge \operatorname{Re} u \in W^{1,2}(\Omega)$. Then $P(\frac{1}{\beta} \operatorname{Tr} u) = \frac{1}{\beta} \operatorname{Tr} w$ and, moreover, $\operatorname{Re} \mathfrak{b}(u - w, w) = 0$. Also $\mathfrak{b}$ is accretive since $c_0 \geq 0$. Here we need the lower bound for c_0. Hence C is invariant under the semigroup generated by $-M_\beta \mathcal{N} M_\beta$ by [AE2] Proposition 2.9.

Step 3 Let $t > 0$ and $\varphi \in L_2(\Gamma, \mathbb{R})$ with $\varphi \leq 1$. Then $\varphi \leq \|\beta\|_\infty \frac{1}{\beta}$. Hence by the above $e^{-tM_\beta \mathcal{N} M_\beta}(\frac{1}{\|\beta\|_\infty}\varphi) \in C$ and $e^{-tM_\beta \mathcal{N} M_\beta}\varphi \leq \|\beta\|_\infty \frac{1}{\beta} \leq \frac{\|\beta\|_\infty}{\operatorname{ess\,inf} \beta}$. So the semigroup generated by $-M_\beta \mathcal{N} M_\beta$ extends to a bounded semigroup on $L_\infty(\Gamma)$ and $\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{\infty \to \infty} \leq \frac{\|\beta\|_\infty}{\operatorname{ess\,inf} \beta}$. By duality the semigroup generated by $-M_\beta \mathcal{N} M_\beta$ extends to a bounded semigroup on $L_1(\Gamma)$ and $\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{1 \to 1} \leq \frac{\|\beta\|_\infty}{\operatorname{ess\,inf} \beta}$. By interpolation the semigroup $(e^{-tM_\beta \mathcal{N} M_\beta})_{t>0}$ extends consistently to a semigroup on $L_p(\Gamma)$ for all $p \in [1, \infty]$.

Step 4 This step is inspired by the proof of Theorem 2.6 in [EO1]. First suppose that $d \geq 3$. By a compactness argument the norm on $W^{1,2}(\Omega)$ is equivalent to $u \mapsto (\mathfrak{b}(u,u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)})^{1/2}$. By Theorem 2.4.2 in [Neč], the trace Tr is a bounded operator from $W^{1,2}(\Omega)$ into $L_s(\Gamma)$, where $s = \frac{2(d-1)}{d-2}$. Hence there exists a $c > 0$ such that

$$\|\operatorname{Tr} u\|^2_{L_s(\Gamma)} \leq c \left(\mathfrak{b}(u,u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)} \right)$$

for all $u \in W^{1,2}(\Omega)$. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Since $e^{-tM_\beta \mathcal{N} M_\beta}\varphi \in D(\widetilde{\mathcal{N}})$, there exists a $u \in W^{1,2}(\Omega)$ such that $\frac{1}{\beta} \operatorname{Tr} u = e^{-tM_\beta \mathcal{N} M_\beta}\varphi$ and $\mathfrak{b}(u,v) = (\widetilde{\mathcal{N}} e^{-tM_\beta \mathcal{N} M_\beta}\varphi, \frac{1}{\beta} \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. Choose $v = u$. Then

$$\begin{aligned}
\|e^{-tM_\beta \mathcal{N} M_\beta}\varphi\|^2_{L_s(\Gamma)} &= \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_s(\Gamma)} \leq \frac{1}{(\operatorname{ess\,inf} \beta)^2} \|\operatorname{Tr} u\|^2_{L_s(\Gamma)} \\
&\leq \frac{c}{(\operatorname{ess\,inf} \beta)^2} \left(\mathfrak{b}(u,u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)} \right) \\
&= \frac{c}{(\operatorname{ess\,inf} \beta)^2} \left((\widetilde{\mathcal{N}} e^{-t\widetilde{\mathcal{N}}}\varphi, e^{-t\widetilde{\mathcal{N}}}\varphi)_{L_2(\Gamma)} + \|e^{-t\widetilde{\mathcal{N}}}\varphi\|^2_{L_2(\Gamma)} \right) \\
&\leq \frac{c}{(\operatorname{ess\,inf} \beta)^2} \left(\frac{1}{e\,t} + 1 \right) \|\varphi\|^2_{L_2(\Gamma)}.
\end{aligned}$$

So $\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{L_2(\Gamma) \to L_s(\Gamma)} \leq \frac{2\sqrt{c}}{\operatorname{ess\,inf} \beta} t^{-1/2}$ if $t \in (0,1]$. Since the semigroup is bounded on $L_\infty(\Gamma)$ and on $L_1(\Gamma)$, one can extrapolate using [Cou] to obtain a $c_1 > 0$ such that

$$\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{L_1(\Gamma) \to L_\infty(\Gamma)} \leq c_1 \, t^{-(d-1)}$$

for all $t \in (0,1]$. Then by interpolation the bounds (7) follow.

Next suppose $d = 2$. Fix $s \in (2, \infty)$. Then it follows from (8) in [EO1] that there exists a $c_2 > 0$ such that

$$\|\varphi\|_{L_s(\Gamma)} \leq c_2 \, \|\varphi\|_{H^{1/2}(\Gamma)}^{1-\theta} \, \|\varphi\|_{L_2(\Gamma)}^{\theta}$$

for all $\varphi \in H^{1/2}(\Gamma)$, where $\theta = 2/s$. The trace is bounded from $W^{1,2}(\Omega)$ into $H^{1/2}(\Gamma)$ by [McL] Theorem 3.37. Hence there exists a $c_3 \geq 1$ such that $\|\operatorname{Tr} u\|_{H^{1/2}(\Gamma)}^2 \leq c_3 \Big(\mathfrak{b}(u,u) + \|\frac{1}{\beta} \operatorname{Tr} u\|_{L_2(\Gamma)}^2 \Big)$ for all $u \in W^{1,2}(\Omega)$. Then

$$\|\operatorname{Tr} u\|_{L_s(\Gamma)} \leq c_2 \, c_3 \left(\mathfrak{b}(u,u) + \|\frac{1}{\beta} \operatorname{Tr} u\|_{L_2(\Gamma)}^2 \right)^{(1-\theta)/2} \|\operatorname{Tr} u\|_{L_2(\Gamma)}^{\theta}$$

for all $u \in W^{1,2}(\Omega)$. Arguing as above it follows that there exists a $c_4 > 0$ such that

$$\|e^{-t M_\beta \mathcal{N} M_\beta}\|_{L_2(\Gamma) \to L_s(\Gamma)} \leq c_4 \, t^{-(\frac{1}{2} - \frac{1}{s})}$$

for all $t \in (0,1]$. Then the bounds (7) follow as before by extrapolation and interpolation.

Step 5 Now we consider Case (II), so we do not assume that $c_0 \geq 0$. Clearly the operator $M_\beta \, \mathcal{N} \, M_\beta$ is self-adjoint and lower bounded. There exists a $\lambda > 0$ such that $c_0 + \lambda \, \mathbb{1}_\Omega \geq \mathbb{1}_\Omega$. Let $\mathcal{N}_0$ be the Dirichlet-to-Neumann operator obtained with c_0 replaced by $c_0 + \lambda \, \mathbb{1}_\Omega$. By [EO2] Corollary 5.6 and Proposition 5.5(d) there exists a bounded self-adjoint operator $Q \colon L_2(\Gamma) \to L_2(\Gamma)$ such that $\mathcal{N} = \mathcal{N}_0 + Q$ and, moreover, for all $p \in [1, \infty]$ the operator Q is consistent with a bounded operator from $L_p(\Gamma)$ into $L_p(\Gamma)$. Then $M_\beta \, \mathcal{N} \, M_\beta = M_\beta \, \mathcal{N}_0 \, M_\beta + M_\beta \, Q \, M_\beta$. The operator $M_\beta \, Q \, M_\beta$ is consistent with a bounded operator from $L_p(\Gamma)$ into $L_p(\Gamma)$ for all $p \in [1, \infty]$. By standard perturbation theory the semigroup generated by $-M_\beta \, \mathcal{N} \, M_\beta$ extends consistently to a semigroup on $L_p(\Gamma)$ for all $p \in [1, \infty]$. By [AE3] Proposition 3.1(a) the semigroup $(e^{-t M_\beta \mathcal{N} M_\beta})_{t>0}$ is again ultracontractive, with the same ultracontractivity exponent. Then by interpolation the bounds (7) follow.

Step 6 It remains to identify in both cases the generator of the semigroup consistent with $(e^{-t M_\beta \mathcal{N} M_\beta})_{t>0}$ on $L_p(\Gamma)$. The semigroup is a C_0-semigroup if $p \in [1, \infty)$ and it is continuous with respect to the weak*-topology if $p = \infty$. Let A_p denote the generator for all $p \in [1, \infty]$. Then $(A_p)^* = A_{p'}$ for all $p \in [1, \infty]$, where p' is the dual exponent. Let $p \in [2, \infty]$. Since Γ has finite measure, it follows that $A_p \subset A_2$. Let $\varphi \in D(A_p)$. Then $\varphi \in D(A_2) \cap L_p(\Gamma)$ and $A_2 \varphi = A_p \varphi \in L_p(\Gamma)$. Now $A_2 = M_\beta \, \mathcal{N} \, M_\beta$. Hence $\beta \, \varphi \in D(\mathcal{N}) \cap L_p(\Gamma)$ and $\mathcal{N}(\beta \, \varphi) = \beta^{-1} \, A_2 \varphi \in L_p(\Gamma)$. Therefore $\beta \, \varphi \in D(\mathcal{N}_p)$ and $\mathcal{N}_p(\beta \, \varphi) = \mathcal{N}(\beta \, \varphi)$. Consequently $\varphi \in D(M_\beta \, \mathcal{N}_p \, M_\beta)$ and $M_\beta \, \mathcal{N}_p \, M_\beta \varphi = \beta \, \mathcal{N}_p(\beta \, \varphi) = \beta \, \mathcal{N}(\beta \, \varphi) = M_\beta \, \mathcal{N} \, M_\beta \varphi = A_2 \varphi = A_p \varphi$. So $A_p \subset M_\beta \, \mathcal{N}_p \, M_\beta$. Similarly $M_\beta \, \mathcal{N}_p \, M_\beta \subset A_p$, so $A_p = M_\beta \, \mathcal{N}_p \, M_\beta$.

Finally, in $p \in [1, 2)$, then $A_p = (A_{p'})^* = (M_\beta \, \mathcal{N}_{p'} \, M_\beta)^* = M_\beta \, \mathcal{N}_p \, M_\beta$. □

Now we consider the multiplicative perturbation of $\mathcal{N}$.

Proposition 3.3. *Suppose*

(I) $c_0 \geq 0$ *or,*

(II) *there exists a* $\kappa > 0$ *such that* Ω *is of class* $C^{1,\kappa}$ *and the principal coefficients* c_{kl} *are uniformly Hölder continuous of order* κ.

Then one has the following.

(a) *The operator* $-\beta\mathcal{N}$ *generates a holomorphic* C_0*-semigroup on* $L_2(\Gamma)$ *with angle* $\frac{\pi}{2}$.

(b) *The semigroup generated by* $-\beta\mathcal{N}$ *extends consistently to a semigroup on* $L_p(\Gamma)$ *for all* $p \in [1,\infty]$, *which is a* C_0*-semigroup if* $p \in [1,\infty)$ *with generator* $-\beta\mathcal{N}_p$.

(c) *There exist* $c, \omega > 0$ *such that*

$$\|e^{-t\beta\mathcal{N}}\|_{L_p(\Gamma)\to L_q(\Gamma)} \leq c\, t^{-(d-1)(\frac{1}{p}-\frac{1}{q})}\, e^{\omega t}$$

for all $t > 0$ *and* $p, q \in [1,\infty]$ *with* $p \leq q$.

Proof. Define $E \colon L_2(\Gamma) \to L_2(\Gamma)$ by $E\varphi = \beta\,\varphi$. Then E is a topological isomorphism. Consider the operator $M_\beta\,\mathcal{N}\,M_\beta$ in Proposition 3.2. Then $\beta^2\,\mathcal{N} = E\,M_\beta\,\mathcal{N}\,M_\beta\,E^{-1}$ is the minus-generator of a holomorphic C_0-semigroup S in $L_2(\Gamma)$. Since also E extends consistently to an topological isomorphism from $L_p(\Gamma)$ onto $L_p(\Gamma)$ for all $p \in [1,\infty]$, all properties for the operator $M_\beta\,\mathcal{N}\,M_\beta$ in Proposition 3.2 carry over to the operator $\beta^2\,\mathcal{N}$, with a different value for the constant c.

Finally replace β by $\sqrt{\beta}$. □

The semigroup generated by $-\beta\mathcal{N}$ is smoothing.

Proposition 3.4. *Suppose*

(I) $c_0 \geq 0$ *or,*

(II) *there exists a* $\kappa > 0$ *such that* Ω *is of class* $C^{1,\kappa}$ *and the principal coefficients* c_{kl} *are uniformly Hölder continuous of order* κ.

Then one has the following.

(a) *Let* $p \in (d-1,\infty)$. *Then* $D(\mathcal{N}_p) \subset C(\Gamma)$.

(b) *Let* $p, q \in [1,\infty]$ *with* $p < q$ *and* $\frac{1}{p} - \frac{1}{q} < \frac{1}{d-1}$. *Then* $D(\mathcal{N}_p) \subset L_q(\Gamma)$.

(c) *Let* $p, q \in [1,\infty]$ *with* $p < q$. *Let* $\varphi \in D(\mathcal{N}_p)$ *and suppose that* $\mathcal{N}_p\varphi \in L_q(\Gamma)$. *Then* $\varphi \in L_q(\Gamma)$.

(d) *If* $t > 0$, *then* $e^{-t\beta\mathcal{N}} L_2(\Gamma) \subset C(\Gamma)$.

Proof. '(a)'. By Proposition 3.2(c) applied to $\beta = \mathbb{1}_\Gamma$ there are $c_1, \omega_1 > 0$ such that $\|e^{-t\mathcal{N}}\|_{L_\infty(\Gamma) \to L_\infty(\Gamma)} \leq c_1 \, e^{\omega_1 t}$ for all $t > 0$. By [EW] Theorem 5.5 and the remark following it, there exist $c_2, \omega_2 > 0$ and $\nu \in (0,1)$ such that $e^{-t\mathcal{N}}$ maps $L_2(\Gamma)$ into $C^\nu(\Gamma)$ and

$$\|e^{-t\mathcal{N}}\|_{L_2(\Gamma) \to C^\nu(\Gamma)} \leq c_2 \, t^{-\frac{(d-1)}{2}} \, t^{-2\nu} \, e^{\omega_2 t}$$

for all $t > 0$. (The exponent -2ν can be replaced by $-\nu$ if $d \geq 3$.) Let $p \in (2, \infty)$. Then by interpolation the operator $e^{-t\mathcal{N}}$ is bounded from $L_p(\Gamma)$ into $C^{2\nu/p}(\Gamma)$ with norm

$$\|e^{-t\mathcal{N}}\|_{L_p(\Gamma) \to C^{2\nu/p}(\Gamma)} \leq c_3 \, t^{-\frac{(d-1)}{p}} \, t^{-\frac{4\nu}{p}} \, e^{\omega_3 t}$$

for all $t > 0$, where $c_3 = c_1 + c_2$ and $\omega_3 = \omega_1 + \omega_2$. Now choose $p = d - 1 + 5\nu$. Then

$$(\mathcal{N}_p + (\omega_3 + 1) \, I)^{-1} = \int_0^\infty e^{-\omega_3 t} \, e^{-t} \, e^{-t\mathcal{N}_p} \, dt$$

maps $L_p(\Gamma)$ into $C^{2\nu/p}(\Gamma)$. In particular $D(\mathcal{N}_p) \subset C^{2\nu/p}(\Gamma) \subset C(\Gamma)$.

'(b)'. The proof is similar, using the bounds of Proposition 3.3(c).

'(c)'. If $\frac{1}{p} - \frac{1}{q} < \frac{1}{d-1}$, then it follows from Statement (b) that $\varphi \in D(\mathcal{N}_p) \cap L_q(\Gamma)$ and by assumption $\mathcal{N}_p \varphi \in L_q(\Gamma)$. So $\varphi \in D(\mathcal{N}_q)$. Now use induction.

'(d)'. Choose $p = d$. Then ultracontractivity and holomorphy on $L_p(\Gamma)$ yield

$$e^{-t\beta\mathcal{N}} L_2(\Gamma) \subset e^{-\frac{t}{2}\beta\mathcal{N}_p} L_p(\Gamma) \subset D(\beta\mathcal{N}_p) = D(\mathcal{N}_p) \subset C(\Gamma)$$

for all $t > 0$. □

For the remainder of this paper we assume Case (II), that is there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ.

Let $C^{0,1}(\Gamma)$ denote the space of Lipschitz continuous functions on Γ. It is endowed with the norm

$$\|g\|_{C^{0,1}(\Gamma)} = \|g\|_{L_\infty(\Gamma)} + \sup_{z,w \in \Gamma, \, z \neq w} \frac{|g(z) - g(w)|}{|z - w|}.$$

For all $g \in C^{0,1}(\Gamma)$ we use the notation $\operatorname{Lip}_\Gamma(g) = \sup_{z,w \in \Gamma, \, z \neq w} \frac{|g(z) - g(w)|}{|z-w|}$. It has been proved in [EO2] Theorem 7.3 that for all $p \in (1, \infty)$ there exists a $c > 0$ such that

$$\|[\mathcal{N}, M_g]\|_{L_p(\Gamma) \to L_p(\Gamma)} \leq c \operatorname{Lip}_\Gamma(g)$$

for all $g \in C^{0,1}(\Gamma)$. These bounds carry over to commutator estimates for the operator $\beta\mathcal{N}$.

Proposition 3.5. *For all $p \in (1, \infty)$ there exists a $c > 0$ such that*

$$\|[\beta\mathcal{N}, M_g]\|_{L_p(\Gamma) \to L_p(\Gamma)} \leq c \operatorname{Lip}_\Gamma(g)$$

for all $g \in C^{0,1}(\Gamma)$.

Proof. Let $g \in C^{0,1}(\Gamma)$. Then

$$[\beta\mathcal{N}, M_g] = M_\beta\,[\mathcal{N}, M_g].$$

So $\|[\beta\mathcal{N}, M_g]\|_{p\to p} \le \|\beta\|_\infty \, \|[\mathcal{N}, M_g]\|_{p\to p}$ and the result follows from [EO2] Theorem 7.3. □

Let $K_\mathcal{N}$ and $K_{\beta\mathcal{N}}$ denote the Schwartz kernel of $\mathcal{N}$ and $\beta\mathcal{N}$. Then $K_{\beta\mathcal{N}}(z,w) = \beta(z)\,K_\mathcal{N}(z,w)$ for all $z,w \in \Gamma$ with $z \neq w$. It follows from [EO2] Proposition 6.5 that there exists a $c > 0$ such that

$$|K_\mathcal{N}(z,w)| \le \frac{c}{|z-w|^d}$$

for all $z,w \in \Gamma$ with $z \neq w$. Consequently one has the next bounds.

Proposition 3.6. *There exists a $c > 0$ such that*

$$|K_{\beta\mathcal{N}}(z,w)| \le \frac{c}{|z-w|^d}$$

for all $z,w \in \Gamma$ with $z \neq w$.

It follows from Proposition 3.3 that for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ the operator $e^{-z\beta\mathcal{N}}$ on $L_2(\Gamma)$ has a kernel $K_z \in L_\infty(\Gamma \times \Gamma)$.

Proposition 3.7. *There exist $c, \omega > 0$ such that*

$$|K_t(w_1,w_2)| \le \frac{c\,t^{-(d-1)}\,e^{\omega t}}{\Big(1 + \dfrac{|w_1 - w_2|}{t}\Big)^d}$$

for all $t > 0$ and $w_1, w_2 \in \Gamma$.

Proof. This follows as in [EO2] Lemma 8.4 and the argument in Section 4 of [EO1]. For the latter, see also [EO2] pages 4270–4272. □

Via an iteration argument the bounds can be improved to the right half of the complex plane. For the convenience of the reader we state the full list of conditions and notation.

Theorem 3.8. *Let $\kappa \in (0,1)$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and connected set of class $C^{1,\kappa}$. Write $\Gamma = \partial\Omega$. For all $k,l \in \{1,\ldots,d\}$ let $c_{kl} \in C^\kappa(\Omega,\mathbb{R})$ and let $c_0 \colon \Omega \to \mathbb{R}$ be a bounded measurable function. Suppose that $c_{kl} = c_{lk}$ for all $k,l \in \{1,\ldots,d\}$. Further, let $\beta \colon \Gamma \to (0,\infty)$ be a bounded measurable function such that $\operatorname{ess\,inf} \beta > 0$. We assume that there exists a $\mu > 0$ such that $\sum_{k,l=1}^d c_{kl}(x)\,\xi_k\,\overline{\xi_l} \ge \mu\,|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Let $\mathfrak{b}$ be the elliptic form as in (4). Suppose that $0 \notin \sigma(B_2^D)$, where B_2^D is the operator associated with the form $\mathfrak{b}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Let $\mathcal{N}$ be the Dirichlet-to-Neumann operator associated with $(\mathfrak{b}, \operatorname{Tr})$.*

Then the operator $-\beta\mathcal{N}$ is the generator of a C_0-semigroup S which is holomorphic with angle $\frac{\pi}{2}$. Moreover, S has a kernel K and for all $\theta \in (0, \frac{\pi}{2})$ there are $c, \omega > 0$ such that

$$|K_z(w_1, w_2)| \leq \frac{c\,|z|^{-(d-1)}\,e^{\omega|z|}}{\Big(1 + \dfrac{|w_1 - w_2|}{|z|}\Big)^d}$$

for all $z \in \mathbb{C} \setminus \{0\}$ and $w_1, w_2 \in \Gamma$ with $|\arg z| \leq \theta$.

Proof. This follows from the previous three propositions as in [EO3] Lemma 3.1 and Theorem 3.2. □

Corollary 3.9. *Adopt the notation and assumptions as in Theorem 3.8. For all $p \in [1, \infty)$ the semigroup $(e^{-t\beta\mathcal{N}})_{t>0}$ extends consistently to a holomorphic semigroup on $L_p(\Gamma)$ with angle $\frac{\pi}{2}$.*

Corollary 3.10. *Adopt the notation and assumptions as in Theorem 3.8. For all $p, r \in (1, \infty)$ the operator $\beta\mathcal{N}_p$ has maximal L_r-regularity on $L_p(\Gamma)$.*

Proof. The operator $-\beta\mathcal{N}$ generates a holomorphic C_0-semigroup on $L_2(\Gamma)$ and this semigroup has a kernel with Poisson bounds by Proposition 3.7. Then the statement follows from Hieber–Prüss [HP] Theorem 3.1. □

Finally we consider the space $C(\Gamma)$. In order to obtain optimal results, without any continuity requirement on β we need the concept of sectorial operators and a maximal operator on $L_\infty(\Gamma)$.

In general, let A be an operator in a Banach space X and let $\alpha \in [0, \pi)$. Then we say that A **is sectorial of angle** α if for all $\theta \in (\alpha, \pi)$ there exist $M, \omega > 0$ such that $\sigma(A + \omega I) \subset \Sigma_\theta$ and

$$\|(A + (\omega + \lambda)I)^{-1}\| \leq \frac{M}{|\lambda|}$$

for all $\lambda \in \mathbb{C}$ with $-\lambda \notin \Sigma_\theta$.

Define the operator $\mathcal{N}_{\infty m} \colon D(\mathcal{N}_{\infty m}) \to L_\infty(\Gamma)$ by

$$D(\mathcal{N}_{\infty m}) = \{\varphi \in D(\mathcal{N}) : \mathcal{N}\varphi \in L_\infty(\Gamma)\}$$

and $\mathcal{N}_{\infty m} = \mathcal{N}|_{D(\mathcal{N}_{\infty m})}$. It follows from Proposition 3.4(c) that $D(\mathcal{N}_{\infty m}) \subset L_\infty(\Gamma)$. We consider $\mathcal{N}_{\infty m}$ as a non-densely defined operator in $L_\infty(\Gamma)$, provided with the norm topology. It is easy to verify that $\mathcal{N}_{\infty m}$ is a closed operator. Moreover, Proposition 3.4(a) gives $D(\mathcal{N}_{\infty m}) \subset C(\Gamma)$.

Lemma 3.11. *Adopt the notation and assumptions as in Theorem 3.8.*

(a) *If $\lambda \in \rho(-\beta\mathcal{N})$, then $\lambda \in \rho(-\beta\mathcal{N}_{\infty m})$ and $(\beta\mathcal{N}_{\infty m} + \lambda I)^{-1} = (\beta\mathcal{N} + \lambda I)^{-1}|_{L_\infty(\Gamma)}$.*

(b) *The operator $\beta\mathcal{N}_{\infty m}$ is sectorial of angle* 0.

(c) $D(\mathcal{N}_{\infty m})$ *is dense in* $C(\Gamma)$.

Proof. '(a)'. Easy.

'(b)'. This follows from the Poisson kernel bounds of Theorem 3.8.

'(c)'. Let $\mathcal{N}_c$ be the part of $\mathcal{N}$ in $C(\Gamma)$. Then $\mathcal{N}_c \subset \mathcal{N}_{\infty m}$. Moreover, $-\mathcal{N}_c$ is the generator of a C_0-semigroup by [EO3] Proposition 2.3. Therefore $D(\mathcal{N}_c)$ is dense in $C(\Gamma)$. Then also $D(\mathcal{N}_{\infty m})$ is dense in $C(\Gamma)$. □

Recall that we do not require β to be continuous. By Proposition 3.4(d) the semigroup $(e^{-t\beta\mathcal{N}})_{t>0}$ leaves the space $C(\Gamma)$ invariant.

Corollary 3.12. *Adopt the notation and assumptions as in Theorem 3.8. Define* $T_t = e^{-t\beta\mathcal{N}}|_{C(\Gamma)} \colon C(\Gamma) \to C(\Gamma)$ *for all* $t > 0$. *Then* T *is a* C_0*-semigroup which is holomorphic with angle* $\frac{\pi}{2}$.

Proof. This follows from Lemma 3.11(b), Lemma 3.11(c) and [ABHN] Remark 3.7.13. □

4 The operator in L_p

We return to the operator $\mathbb{A}$ with dynamical boundary conditions as introduced in Section 2. Under the smoothness assumptions as in Case (II) in Section 3 we show that for all $p \in (1,\infty)$ the operator $\mathbb{A}$ is consistent with an operator $\mathbb{A}_p$ on L_p such that $-\mathbb{A}_p$ is the generator of a C_0-semigroup which is holomorphic on the right half-plane and $\mathbb{A}_p$ has maximal L_r-regularity for all $r \in (1,\infty)$.

We extend the definition of $\mathbb{L}_2$. Define

$$\mathbb{L}_p = L_p(\Omega) \times L_p(\Gamma)$$

for all $p \in [1,\infty]$, with norm

$$\|(u,\varphi)\|_{\mathbb{L}_p}^p = \int_\Omega |u|^p + \int_\Gamma |\varphi|^p \, \frac{d\sigma}{\beta}$$

and obvious modification if $p = \infty$.

The main theorem of this section is as follows. In Corollary 5.6 we consider the case $p = 1$. For the convenience of the reader we repeat the standing assumptions.

Theorem 4.1. *Let* $\kappa \in (0,1)$. *Let* $\Omega \subset \mathbb{R}^d$ *be an open, bounded and connected set of class* $C^{1,\kappa}$. *Write* $\Gamma = \partial\Omega$. *For all* $k,l \in \{1,\ldots,d\}$ *let* $c_{kl} \in C^\kappa(\Omega,\mathbb{R})$ *and let* $c_0 \colon \Omega \to \mathbb{R}$ *be a bounded measurable function. Suppose that* $c_{kl} = c_{lk}$ *for all* $k,l \in \{1,\ldots,d\}$. *Further, let* $\alpha \colon \Gamma \to \mathbb{C}$ *be a bounded measurable function and let* $\beta \colon \Gamma \to (0,\infty)$ *be a bounded measurable function with* $\operatorname{ess\,inf} \beta > 0$. *We assume that there exists a* $\mu > 0$ *such that* $\sum_{k,l=1}^d c_{kl}(x)\, \xi_k \, \overline{\xi_l} \geq \mu\, |\xi|^2$ *for all* $x \in \Omega$ *and* $\xi \in \mathbb{C}^d$. *Let* $\mathbb{A}$ *be the associated variational operator in* $\mathbb{L}_2$ *as in Section 2. Then for all* $p \in (1,\infty)$ *the semigroup generated by* $-\mathbb{A}$ *extends consistently to a* C_0*-semigroup on* $\mathbb{L}_p$ *which is holomorphic with angle* $\frac{\pi}{2}$. *Moreover, its generator has maximal* L_r*-regularity on* $\mathbb{L}_p$ *for all* $r \in (1,\infty)$.

The proof requires quite some preparation. The main problem to circumvent is that we cannot apply the divergence theorem since the principal coefficients are not Lipschitz continuous. Adopt the notation and assumptions as in Theorem 4.1. We use the notation as in Section 2. As in Section 3 define the form $\mathfrak{b}_D \colon W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{C}$ by $\mathfrak{b}_D = \mathfrak{b}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Then $\mathfrak{b}_D$ is a closed sectorial form in $L_2(\Omega)$. Let B_2^D be the operator associated to $\mathfrak{b}_D$. By Lemma 2.2 we may assume that $0 \notin \sigma(B_2^D)$. The operator $-B_2^D$ generates a holomorphic C_0-semigroup $S^{(2)D}$ on $L_2(\Omega)$ with angle $\frac{\pi}{2}$. Moreover, for all $p \in [1,\infty]$ the semigroup $S^{(2)D}$ extends consistently to a semigroup $S^{(p)D}$ on $L_p(\Omega)$ with angle $\frac{\pi}{2}$ by [Ouh1] Theorem 3.1(1). Moreover, $S^{(p)D}$ is a C_0-semigroup for all $p \in [1,\infty)$. We denote by $-B_p^D$ the generator of $S^{(p)D}$. Then B_p^D is strongly consistent with B_2^D by Lemma 3.1(b). Since the semigroup $S^{(2)D}$ has Gaussian kernel bounds by [AE1] Theorem 4.4, one deduces from [Kun] Theorem 1.1 that $\sigma(B_p^D) = \sigma(B_2^D)$. So B_p^D is invertible. Then also $(B_p^D)^{-1}$ is consistent with $(B_2^D)^{-1}$.

In addition we need a harmonic lifting (also called harmonic extension). Since $0 \notin \sigma(B_2^D)$ one can define $\gamma \colon \operatorname{Tr} W^{1,2}(\Gamma) \to W^{1,2}(\Omega)$ by $\gamma(\varphi) = u$, where $u \in W^{1,2}(\Omega)$ is such that $\operatorname{Tr} u = \varphi$ and $\mathcal{B}u = 0$. (So u is $\mathcal{B}$-harmonic.) Then by [EO2] Proposition 5.5 there exists an operator $\mathcal{H} \colon L_1(\Gamma) \to C(\Omega) \cap L_1(\Omega)$ such that

$$\begin{aligned}
&\mathcal{H}|_{\operatorname{Tr} W^{1,2}(\Omega)} = \gamma, \\
&\mathcal{H}(L_p(\Gamma)) \subset L_p(\Omega) \text{ for all } p \in [1,\infty], \text{ and} \\
&\mathcal{H}|_{L_p(\Gamma)} \colon L_p(\Gamma) \to L_p(\Omega) \text{ is continuous for all } p \in [1,\infty].
\end{aligned}$$

We write $\gamma_p = \mathcal{H}|_{L_p(\Gamma)} \colon L_p(\Gamma) \to L_p(\Omega)$ for all $p \in [1,\infty]$.

There is a remarkable relation between the elliptic operator B_p^D with Dirichlet boundary conditions, the harmonic lifting γ and the (weak) co-normal derivative. We denote by $(\nu_1, \ldots, \nu_d)$ the outer normal on Γ.

Proposition 4.2.

(a) $D(B_2^D) \subset D(\partial_\nu^C)$.

(b) *If* $p \in [2,\infty)$, *then* $\gamma_{p'}^* = -\partial_\nu^C \, (B_p^D)^{-1}$.

(c) *If* $p \in (d+2\kappa,\infty)$, *then* $D(B_p^D) \subset C^{1+2\kappa/p}(\Omega)$.

(d) *If* $p \in (d+2\kappa,\infty)$ *and* $u \in D(B_p^D)$, *then*

$$\partial_\nu^C u = \sum_{k,l=1}^d \nu_k \, (c_{kl} \, \partial_l u)|_\Gamma, \tag{8}$$

where we extend an element of $C^{2\kappa/p}(\Omega)$ *to* $\overline{\Omega}$ *by continuity. In particular,* $\partial_\nu^C u \in C(\Gamma)$.

Proof. Statement (c) follows from [EO2] Proposition 4.3. If $p \in (d+2\kappa,\infty)$, then $D(B_p^D) \subset D(\partial_\nu^C)$ by [EO2] Proposition 5.3 and Statement (d) follows by the same proposition.

Next choose $p = d + 3\kappa$. Let $\varphi \in \operatorname{Tr} W^{1,2}(\Omega)$ and $u \in L_p(\Omega)$. Then

$$-(u, \gamma\varphi)_{L_2(\Omega)} = (\partial_\nu^C \, (B_p^D)^{-1} u, \varphi)_{L_2(\Gamma)}$$

by [EO2] Lemma 5.4. Replacing u by $B_p^D u$ gives

$$-(B_p^D u, \gamma \operatorname{Tr} v)_{L_2(\Omega)} = (\partial_\nu^C u, \operatorname{Tr} v)_{L_2(\Gamma)} = \mathfrak{b}(u,v) - (B_2^D u, v)_{L_2(\Omega)}$$

for all $u \in D(B_p^D)$ and $v \in W^{1,2}(\Omega)$. Now let $u \in D(B_2^D)$. Since $D(B_p^D)$ is a core for B_2^D by [ER] Lemma 3.8, there are $u_1, u_2, \ldots \in D(B_p^D)$ such that $\lim u_n = u$ and $\lim B_p^D u_n = B_2^D u$ in $L_2(\Omega)$. Then the ellipticity of $\mathfrak{b}$ gives $\lim u_n = u$ in $W^{1,2}(\Omega)$. Let $v \in W^{1,2}(\Omega)$. Then

$$-(B_2^D u_n, \gamma_2 \operatorname{Tr} v)_{L_2(\Omega)} = \mathfrak{b}(u_n, v) - (B_2^D u_n, v)_{L_2(\Omega)}$$

for all $n \in \mathbb{N}$. Using the continuity of γ_2 and taking the limit $n \to \infty$ gives

$$(-\gamma_2^* B_2^D u, \operatorname{Tr} v)_{L_2(\Omega)} = -(B_2^D u, \gamma_2 \operatorname{Tr} v)_{L_2(\Omega)} = \mathfrak{b}(u,v) - (B_2^D u, v)_{L_2(\Omega)}.$$

Then by definition of the weak co-normal derivative one deduces that $u \in D(\partial_\nu^C)$ and $\partial_\nu^C u = -\gamma_2^* B_2^D u$. This proves Statement (a) and also Statement (b) for the endpoint.

Finally let $p \in (2, \infty)$. If $u \in L_p(\Omega)$, then $\gamma_{p'}^* u = \gamma_2^* u = \partial_\nu^C \, (B_2^D)^{-1} u = \partial_\nu^C \, (B_p^D)^{-1} u$ and Statement (b) follows. □

Proposition 4.2(b) implies that $\partial_\nu^C u = -\gamma_{p'}^* \, B_p^D u$ for all $p \in [2, \infty)$ and $u \in D(B_p^D)$. It is unclear whether $D(B_p^D) \subset D(\partial_\nu^C)$ and $\partial_\nu^C u = -\gamma_{p'}^* \, B_p^D u$ for all $p \in (1,2)$ and $u \in D(B_p^D)$. We circumvent this problem by working with the operator $\gamma_{p'}^* \, B_p^D$ on $D(B_p^D)$. In addition we do not know whether (8) extends to $p \in (1, d]$ or even how to interpretate the right hand side.

For all $p \in [1, \infty)$ define $\widehat{\mathbb{A}}_p \colon D(B_p^D) \times D(\mathcal{N}_p) \to \mathbb{L}_p$ by

$$\widehat{\mathbb{A}}_p = \begin{pmatrix} B_p^D + \gamma_p \, M_\beta \, \gamma_{p'}^* \, B_p^D & -\gamma_p \, M_\beta \, \mathcal{N}_p - \gamma_p \, M_\alpha \\ -M_\beta \, \gamma_{p'}^* \, B_p^D & \beta \mathcal{N}_p + M_\alpha \end{pmatrix}.$$

We need a technical lemma.

Lemma 4.3. *Let $p \in (1, \infty)$. Then one has the following.*

(a) *The operator $\widehat{\mathbb{A}}_p$ is strongly consistent with $\widehat{\mathbb{A}}_2$.*

(b) *The operator $-\widehat{\mathbb{A}}_p$ is the generator of a C_0-semigroup in $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$.*

(c) *The semigroup generated by $-\widehat{\mathbb{A}}_p$ is consistent with the semigroup generated by $-\widehat{\mathbb{A}}_2$.*

(d) *For all $r \in (1, \infty)$ the operator $\widehat{\mathbb{A}}_p$ has maximal L_r-regularity on $\mathbb{L}_p$.*

For the proof of Lemma 4.3 we use an abstract lemma, which is contained in the proof of [EF] Lemma A.4.

Lemma 4.4. *Let A and B be operators in Banach spaces X and Y, respectively. Let $P_1 \colon D(A) \to X$ and $P_2 \colon D(A) \to Y$ be relatively A-bounded with A-bound zero. Let $P_3 \colon D(B) \to Y$ be relatively B-bounded with B-bound zero and $Q \colon D(B) \to X$ be a relatively B-bounded operator. Define*

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \textit{and} \quad \mathcal{P} = \begin{pmatrix} P_1 & Q \\ P_2 & P_3 \end{pmatrix}.$$

Then for all $\varepsilon > 0$ there exists an isomorphism $S \colon X \times Y \to X \times Y$ such that $S\,\mathcal{A}_0\,S^{-1} = \mathcal{A}_0$ and $S\,\mathcal{P}\,S^{-1}$ is relatively $\mathcal{A}_0$-bounded with $\mathcal{A}_0$-bound ε.

Proof of Lemma 4.3. Statement (a) is obvious.

'(b)'. First note that

$$\widehat{\mathbb{A}}_p - \begin{pmatrix} B_p^D & 0 \\ 0 & \beta \mathcal{N}_p \end{pmatrix} = \begin{pmatrix} \gamma_p\, M_\beta\, \gamma_{p'}^*\, B_p^D & -\gamma_p\, M_\beta\, \mathcal{N}_p - \gamma_p\, M_\alpha \\ -M_\beta\, \gamma_{p'}^*\, B_p^D & M_\alpha \end{pmatrix}. \tag{9}$$

The operator $-B_p^D$ generates a C_0-semigroup in $L_p(\Omega)$ which is holomorphic with angle $\frac{\pi}{2}$ and by Corollary 3.9 the operator $-\beta\mathcal{N}_p$ generates a C_0-semigroup in $L_p(\Gamma)$ which is holomorphic with angle $\frac{\pi}{2}$. It follows that

$$\begin{pmatrix} B_p^D & 0 \\ 0 & \beta \mathcal{N}_p \end{pmatrix}$$

is the minus-generator of a C_0-semigroup in $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. We next show that $\gamma_{p'}^*\, B_p^D$ is B_p^D-bounded with relative bound zero.

Let $q = (p+1) \vee (d + 3\kappa)$. It follows as in the proof of [EO2] Proposition 4.3 that there are $c, \omega > 0$ such that $\nabla\, e^{-tB_q^D} u \in C^{2\kappa/q}(\Omega) \subset C(\overline{\Omega})$ for all $u \in L_q(\Omega)$ and

$$\|\nabla\, e^{-tB_q^D}\|_{L_q(\Omega) \to C(\overline{\Omega})} \le c\, t^{-\frac{d}{2q}}\, t^{-\frac{1}{2}}\, e^{\omega t}$$

for all $t > 0$. Then a Laplace transform together with Proposition 4.2(d) gives

$$\begin{aligned}
\|\partial_\nu^C (B_q^D + \lambda\, I)^{-1} u\|_{L_q(\Gamma)} &\le (\sigma(\Gamma))^{1/q}\, \|C\|_\infty\, \|\nabla (B_q^D + \lambda\, I)^{-1} u\|_{C(\overline{\Omega})} \\
&\le c\, (\sigma(\Gamma))^{1/q}\, \|C\|_\infty \int_0^\infty t^{-\frac{d}{2q}}\, t^{-\frac{1}{2}}\; e^{-(\lambda - \omega)t}\, \|u\|_{L_q(\Omega)}\, dt \\
&= c_1\, (\lambda - \omega)^{-(\frac{1}{2} - \frac{d}{2q})}\, \|u\|_{L_q(\Omega)}
\end{aligned} \tag{10}$$

for all $\lambda > \omega$, where $c_1 = c\,(\sigma(\Gamma))^{1/q}\, \Gamma(\frac{1}{2} - \frac{d}{2q})\, \|C\|_\infty$ and $\|C\|_\infty = \sup_{x \in \Omega} \|C(x)\|_{\mathcal{L}(\mathbb{C}^d)}$. Hence if $\lambda > \omega$ and $u \in D(B_q^D)$, then it follows from Proposition 4.2(b) that

$$\|\gamma_{q'}^*\, B_q^D u\|_{L_q(\Gamma)} = \|\partial_\nu^C u\|_{L_q(\Gamma)} \le c_1\, (\lambda - \omega)^{-(\frac{1}{2} - \frac{d}{2q})}\, \|(B_q^D + \lambda\, I) u\|_{L_q(\Omega)}. \tag{11}$$

Next, since $-B_1^D$ generates a C_0-semigroup there are $c_2, \omega_2 > 0$ such that $B_1^D + \lambda I$ is invertible and $\|B_1^D (B_1^D + \lambda I)^{-1}\|_{L_1(\Omega)\to L_1(\Omega)} \leq c_2$ for all $\lambda \geq \omega_2$. Then

$$\|\gamma_{1'}^* B_1^D u\|_{L_1(\Gamma)} \leq \|\gamma_\infty\|_{L_\infty(\Gamma)\to L_\infty(\Omega)} c_2 \|(B_1^D + \lambda I)u\|_{L_1(\Omega)} = c_3 \|(B_1^D + \lambda I)u\|_{L_1(\Omega)} \quad (12)$$

for all $\lambda \geq \omega_2$ and $u \in D(B_1^D)$, where $c_3 = c_2 \|\gamma_\infty\|_{L_\infty(\Gamma)\to L_\infty(\Omega)}$. There exists a $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{1}$. Note that $\theta > 0$ since $p > 1$. Interpolation between (11) and (12) gives

$$\|\gamma_{p'}^* B_p^D u\|_{L_p(\Gamma)} \leq c_1^\theta c_3^{1-\theta} (\lambda - \omega)^{-\delta} \|(B_p^D + \lambda I)u\|_{L_p(\Omega)}$$

for all $u \in D(B_p^D)$ and $\lambda > \omega \vee \omega_2$, where $\delta = \theta(\frac{1}{2} - \frac{d}{2q}) > 0$. Then

$$\|\gamma_{p'}^* B_p^D u\|_{L_p(\Gamma)} \leq c_1^\theta c_3^{1-\theta} (\lambda - \omega)^{-\delta} \|B_p^D u\|_{L_p(\Omega)} + c_1^\theta c_3^{1-\theta} \lambda (\lambda - \omega)^{-\delta} \|u\|_{L_p(\Omega)}$$

for all $u \in D(B_p^D)$ and $\lambda > \omega \vee \omega_2$. So $\gamma_{p'}^* B_p^D$ and hence also $M_\beta \gamma_{p'}^* B_p^D$ and $\gamma_p M_\beta \gamma_{p'}^* B_p^D$ are B_p^D-bounded with relative bound zero.

Using (9) it now follows from Lemma 4.4 and a standard perturbation argument that $-\widehat{\mathbb{A}}_p$ is the generator of a C_0-semigroup on $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. This completes the proof of Statement (b).

'(c)'. This follows from Statement (b) and Lemma 3.1(c).

'(d)'. We use again the perturbation (9). Write $\widehat{\mathbb{A}}_p^{(0)} = \begin{pmatrix} B_p^D & 0 \\ 0 & \beta \mathcal{N}_p \end{pmatrix}$. The operator B_p^D has maximal L_r-regularity by [HP] Example B (since $p \in (1,\infty)$) and it is the minus-generator of a holomorphic semigroup. So together with Corollaries 3.9 and 3.10 the operator $\widehat{\mathbb{A}}_p^{(0)}$ has maximal L_r-regularity on $L_p(\Omega) \times L_p(\Gamma)$ and it is the minus-generator of a holomorphic semigroup. It follows from Lemma 4.4 that for all $\varepsilon > 0$ there exists an isomorphism $S \colon L_p(\Omega) \times L_p(\Gamma) \to L_p(\Omega) \times L_p(\Gamma)$ such that $S \, \widehat{\mathbb{A}}_p^{(0)} S^{-1} = \widehat{\mathbb{A}}_p^{(0)}$ and $S \, (\widehat{\mathbb{A}}_p - \widehat{\mathbb{A}}_p^{(0)}) \, S^{-1}$ is relatively $\widehat{\mathbb{A}}_p^{(0)}$-bounded with $\widehat{\mathbb{A}}_p^{(0)}$-bound ε. If ε is small enough, then [KW] Corollary 2 implies that

$$S \, \widehat{\mathbb{A}}_p \, S^{-1} = \widehat{\mathbb{A}}_p^{(0)} + S \, (\widehat{\mathbb{A}}_p - \widehat{\mathbb{A}}_p^{(0)}) \, S^{-1}$$

has maximal L_r-regularity on $L_p(\Omega) \times L_p(\Gamma)$. But then also $\widehat{\mathbb{A}}_p$ has maximal L_r-regularity first on $L_p(\Omega) \times L_p(\Gamma)$ and then also on $\mathbb{L}_p$, since it has an equivalent norm. □

For all $p \in (1,\infty)$ the operator

$$\begin{pmatrix} I & -\gamma_p \\ 0 & I \end{pmatrix}$$

is an invertible operator from $\mathbb{L}_p$ onto $\mathbb{L}_p$ with inverse

$$\begin{pmatrix} I & \gamma_p \\ 0 & I \end{pmatrix}.$$

We define the operator $\mathbb{A}_p$ in $\mathbb{L}_p$ by

$$\mathbb{A}_p = \begin{pmatrix} I & \gamma_p \\ 0 & I \end{pmatrix} \widehat{\mathbb{A}}_p \begin{pmatrix} I & -\gamma_p \\ 0 & I \end{pmatrix}.$$

A reformulation and extension of Theorem 4.1 is the following theorem.

Theorem 4.5. *Adopt the notation and assumptions as in Theorem 4.1. Let $p \in (1, \infty)$. Then one has the following.*

(a) $\mathbb{A} = \mathbb{A}_2$.

(b) *The operator $\mathbb{A}_p$ is strongly consistent with $\mathbb{A}_2$.*

(c) *The operator $-\mathbb{A}_p$ is the generator of a C_0-semigroup in $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$ and consistent with the semigroup generated by $-\mathbb{A}_2$.*

(d) *For all $r \in (1, \infty)$ the operator $\mathbb{A}_p$ has maximal L_r-regularity on $\mathbb{L}_p$.*

Proof. The proofs of Statements (b), (c) and (d) are obvious.

'(a)'. We first prove that $\mathbb{A}_2 \subset \mathbb{A}$. Let $(u, \varphi) \in D(\mathbb{A}_2)$. Write $(f, \eta) = \mathbb{A}_2(u, \varphi)$. Then $(u - \gamma_2\varphi, \varphi) \in D(\widehat{\mathbb{A}}_2)$ and $\widehat{\mathbb{A}}_2(u - \gamma_2\varphi, \varphi) = (f - \gamma_2\eta, \eta)$. So

$$u - \gamma_2\varphi \in D(B_2^D), \tag{13}$$

$$\varphi \in D(\mathcal{N}_2), \tag{14}$$

$$(B_2^D + \gamma_2 \, M_\beta \, \gamma_2^* \, B_2^D)(u - \gamma_2\varphi) - \gamma_2(\beta \, \mathcal{N}_2\varphi) - \gamma_2(\alpha \, \varphi) = f - \gamma_2\eta, \text{ and} \tag{15}$$

$$(-\beta \, \gamma_2^* \, B_2^D)(u - \gamma_2\varphi) + \beta \, \mathcal{N}_2\varphi + \alpha \, \varphi = \eta. \tag{16}$$

If follows from (14) that $\varphi \in D(\mathcal{N}_2) \subset \mathrm{Tr}\, W^{1,2}(\Omega)$. Hence $\gamma_2\varphi = \gamma\varphi \in W^{1,2}(\Omega)$. Then (13) implies that $u - \gamma\varphi = u - \gamma_2\varphi \in D(B_2^D) \subset W_0^{1,2}(\Omega)$. So $0 = \mathrm{Tr}\,(u - \gamma\varphi) = \mathrm{Tr}\, u - \varphi$ and hence $\mathrm{Tr}\, u = \varphi$. It follows from Proposition 4.2 that $(B_2^D)^{-1}v \in D(\partial_\nu^C)$ and $\partial_\nu^C \, (B_2^D)^{-1}v = -\gamma_2^* v$ for all $v \in L_2(\Omega)$. Since $u - \gamma_2\varphi \in D(B_2^D)$ by (13) one can choose $v = B_2^D(u - \gamma_2\varphi)$ to deduce that $(u - \gamma_2\varphi) \in D(\partial_\nu^C)$ and $\partial_\nu^C(u - \gamma_2\varphi) = -\gamma_2^* \, B_2^D(u - \gamma_2\varphi)$.

Applying γ_2 to (16) and adding (15) gives $B_2^D(u - \gamma\varphi) = f$. Taking the inner product with $\tau \in C_c^\infty(\Omega)$ gives $\mathfrak{b}(u, \tau) = \mathfrak{b}(u - \gamma\varphi, \tau) = (f, \tau)_{L_2(\Omega)}$. So $\mathcal{B}u = f$. Finally, since $\varphi \in D(\mathcal{N}_2)$ it follows that $\gamma\varphi$ has a co-normal derivative and $\partial_\nu^C \gamma\varphi = \mathcal{N}_2\varphi$. Hence u has a co-normal derivative and

$$\eta = \beta \, \partial_\nu^C(u - \gamma_2\varphi) + \beta \, \mathcal{N}_2\varphi + \alpha \, \varphi = \beta \, \partial_\nu^C u + \alpha \mathrm{Tr}\, u.$$

Now it follows from Lemma 2.1 (ii)$\Rightarrow$(i) that $(u, \varphi) \in D(\mathbb{A})$ and $\mathbb{A}(u, \varphi) = (f, \eta)$. So $\mathbb{A}_2 \subset \mathbb{A}$.

We proved that $\mathbb{A}$ is an extension of $\mathbb{A}_2$. Next, both $-\mathbb{A}_2$ and $-\mathbb{A}$ generate a C_0-semigroup. Hence $\mathbb{A} = \mathbb{A}_2$. This completes the proof of the theorem and also of Theorem 4.1. □

5 The operator in the space of continuous functions

In Section 4 we proved that the semigroup generated by $-\mathbb{A}$ extends consistently to a C_0-semigroup on $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. In this section we aim to prove a similar result on the space of continuous function which satisfy a trace condition. Define

$$X_c = \{(u, \varphi) \in C(\overline{\Omega}) \times C(\Gamma) : u|_\Gamma = \varphi\}.$$

Then X_c is naturally isomorphic with $C(\overline{\Omega})$. Let $\mathbb{A}_c$ be the part of $\mathbb{A}$ in X_c. The main theorem of this section is as follows. We emphasise that we do not assume that α or β are continuous.

Theorem 5.1. *Adopt the notation and assumptions as in Theorem 4.1. Then $-\mathbb{A}_c$ is the generator of a C_0-semigroup in X_c which is holomorphic with angle $\frac{\pi}{2}$ and consistent with the semigroup generated by $-\mathbb{A}$.*

The proof requires again some preparation. Throughout the remainder of this section we adopt the notation and assumptions as in Theorem 4.1 and Section 4. Again by Lemma 2.2 we may assume that $0 \notin \sigma(B_2^D)$.

We need a maximal version for B^D on $L_\infty(\Omega)$ similar to $\mathcal{N}_{\infty m}$. Define the operator $B_{\infty m}^D \colon D(B_{\infty m}^D) \to L_\infty(\Omega)$ by

$$D(B_{\infty m}^D) = \{u \in D(B_2^D) : B_2^D u \in L_\infty(\Omega)\}$$

and $B_{\infty m}^D = B_2^D|_{D(B_{\infty m}^D)}$. Then $D(B_{\infty m}^D) \subset C_0(\Omega)$ by [AE4] Corollary 2.10. We consider $B_{\infty m}^D$ as a non-densely defined operator in $L_\infty(\Omega)$. Then $B_{\infty m}^D$ is a closed operator.

Lemma 5.2.

(a) *If $\lambda \in \rho(-B_2^D)$, then $\lambda \in \rho(-B_{\infty m}^D)$ and $(B_{\infty m}^D + \lambda\, I)^{-1} = (B_2^D + \lambda\, I)^{-1}|_{L_\infty(\Omega)}$.*

(b) *The operator $B_{\infty m}^D$ is sectorial of angle 0.*

(c) *$D(B_{\infty m}^D)$ is dense in $C_0(\Omega)$.*

(d) *If $u \in D(B_{\infty m}^D)$, then $u \in D(\partial_\nu^C)$ and $\partial_\nu^C u \in C(\Gamma)$.*

(e) *The operator ∂_ν^C is relatively $B_{\infty m}^D$-bounded with relative $B_{\infty m}^D$-bound zero.*

Proof. '(a)'. Easy.

'(b)'. This follows from the Gaussian kernel bounds for the semigroup generated by $-B_2^D$.

'(c)'. Let B_c^D be the part of B_2^D in $C_0(\Omega)$. Then $B_c^D \subset B_{\infty m}^D$. Moreover, $-B_c^D$ is the generator of a C_0-semigroup by [AE4] Theorem 1.3. Hence $D(B_c^D)$ is dense in $C_0(\Omega)$. Then also $D(B_{\infty m}^D)$ is dense in $C_0(\Omega)$.

'(d)'. See Proposition 4.2.

'(e)'. Let $p = d + 3\kappa$. If $u \in D(B^D_{\infty m})$, then $u \in C_0(\Omega) \cap D(B^D_2) \subset L_p(\Omega) \cap D(B^D_2)$ and $B^D_2 u \in L_\infty(\Omega) \subset L_p(\Omega)$. Hence $u \in D(B^D_p)$. The Gaussian derivative bounds of [EO2] Theorem 3.1(a) give that there exist $c, \omega > 0$ such that

$$\|\nabla e^{-tB^D_p} u\|_{L_\infty(\Omega)} \leq c\, t^{-1/2}\, e^{\omega t}\, \|u\|_{L_\infty(\Omega)}$$

for all $t > 0$ and $u \in L_\infty(\Omega)$. Arguing as in (10) one deduces that there is a $c' > 0$ such that

$$\|\partial^C_\nu\, (B^D_{\infty m} + \lambda\, I)^{-1} u\|_{C(\Gamma)} = \|\partial^C_\nu\, (B^D_p + \lambda\, I)^{-1} u\|_{C(\Gamma)} \leq c'\, (\lambda - \omega)^{-1/2}\, \|u\|_{L_\infty(\Omega)}$$

for all $u \in L_\infty(\Omega)$ and $\lambda > \omega$. So

$$\begin{aligned} \|\partial^C_\nu u\|_{C(\Gamma)} &\leq c'\, (\lambda - \omega)^{-1/2}\, \|(B^D_{\infty m} + \lambda\, I) u\|_{L_\infty(\Omega)} \\ &\leq c'\, (\lambda - \omega)^{-1/2}\, \|B^D_{\infty m} u\|_{L_\infty(\Omega)} + c'\, (\lambda - \omega)^{-1/2}\, \lambda\, \|u\|_{L_\infty(\Omega)} \end{aligned}$$

and the statement follows. □

By Lemma 5.2(d) we can define $\widetilde{\mathbb{A}}_\infty \colon D(B^D_{\infty m}) \times D(\mathcal{N}_{\infty m}) \to L_\infty(\Omega) \times L_\infty(\Gamma)$ by

$$\widetilde{\mathbb{A}}_\infty = \begin{pmatrix} B^D_{\infty m} - \gamma_\infty\, M_\beta\, \partial^C_\nu & -\gamma_\infty\, M_\beta\, \mathcal{N}_{\infty m} - \gamma_\infty\, M_\alpha \\ M_\beta\, \partial^C_\nu & \beta \mathcal{N}_{\infty m} + M_\alpha \end{pmatrix}.$$

We consider $\widetilde{\mathbb{A}}_\infty$ as a non-densely defined closed operator in $L_\infty(\Omega) \times L_\infty(\Gamma)$.

Proposition 5.3.

(a) *The operator $\widetilde{\mathbb{A}}_\infty$ is sectorial of angle 0.*

(b) *$\widetilde{\mathbb{A}}_\infty \subset \widehat{\mathbb{A}}_p$ for all $p \in (1, \infty)$.*

Proof. '(a)'. By Lemma 5.2(b) and Lemma 3.11(b) the operator $\begin{pmatrix} B^D_{\infty m} & 0 \\ 0 & \beta \mathcal{N}_{\infty m} \end{pmatrix}$ is sectorial of angle 0. Then the statement follows from [EF] Lemma A.4 and Lemma 5.2(e).

'(b)'. By Lemma 4.3 it suffices to prove the statement for all $p \in [2, \infty)$. Let $p \in [2, \infty)$. Then $\partial^C_\nu u = -\gamma^*_{p'}\, B^D_p u$ for all $u \in D(B^D_p)$ by Proposition 4.2(b) and the inclusion follows. □

The domain of $\widetilde{\mathbb{A}}_\infty$ is not dense in $L_\infty(\Omega) \times L_\infty(\Gamma)$. We take a suitable restriction. Let $\widehat{\mathbb{A}}_c$ be the part of $\widetilde{\mathbb{A}}_c$ in $C_0(\Omega) \times C(\Gamma)$.

Proposition 5.4.

(a) *$-\widehat{\mathbb{A}}_c$ is the generator of a C_0-semigroup in $C_0(\Omega) \times C(\Gamma)$ which is holomorphic with angle $\frac{\pi}{2}$.*

(b) *$\widehat{\mathbb{A}}_c \subset \widehat{\mathbb{A}}_p$ for all $p \in (1, \infty)$.*

(c) $\widehat{\mathbb{A}}_c$ *is the part of* $\widehat{\mathbb{A}}_2$ *in* $C_0(\Omega) \times C(\Gamma)$.

(d) *The semigroup generated by* $-\widehat{\mathbb{A}}_c$ *is consistent with the semigroup generated by* $-\widehat{\mathbb{A}}_2$.

Proof. '(a)'. It follows from Lemma 5.2(c) and Lemma 3.11(c) that $D(\widetilde{\mathbb{A}}_\infty)$ is dense in $C_0(\Omega) \times C(\Gamma)$. Also the operator $\widetilde{\mathbb{A}}_\infty$ is sectorial with angle $\frac{\pi}{2}$ by Proposition 5.3(a). Now the statement follows from [ABHN] Remark 3.7.13.

'(b) and (c)'. This follows from the definition of $\widehat{\mathbb{A}}_c$ and Proposition 5.3(b).

'(d)'. This follows from Statement (c) and Lemma 3.1(c). □

Now we are able to proof the main theorem in this section.

Proof of Theorem 5.1. Define $\gamma_c \colon C(\Gamma) \to C(\overline{\Omega})$ by $\gamma_c = \gamma_\infty|_{C(\Gamma)}$. So $\gamma_c(\varphi)$ is the classical solution of the Dirichlet problem with boundary data φ. The operator $\begin{pmatrix} I & -\gamma_c \\ 0 & I \end{pmatrix}$ maps X_c onto $C_0(\Omega) \times C(\Gamma)$. Therefore Proposition 5.4(c) implies that

$$\mathbb{A}_c = \begin{pmatrix} I & \gamma_c \\ 0 & I \end{pmatrix} \widehat{\mathbb{A}}_c \begin{pmatrix} I & -\gamma_c \\ 0 & I \end{pmatrix}$$

and the theorem follows from Proposition 5.4(a). □

Since $u \mapsto (u, u|_\Gamma)$ is an isomorphism from $C(\overline{\Omega})$ onto X_c one can reformulate Theorem 5.1. Recall once again that we do not require that α and β are continuous.

Theorem 5.5. *Adopt the notation and assumptions as in Theorem 4.1. Define the operator* A_c *in* $C(\overline{\Omega})$ *by*

$$D(A_c) = \{u \in C(\overline{\Omega}) \cap D(\partial_\nu^C) : \mathcal{B}u \in C(\overline{\Omega}) \text{ and } (\mathcal{B}u)|_\Gamma = \beta\, \partial_\nu^C u + \alpha\, u|_\Gamma \text{ a.e. on } \Gamma\}$$

and $A_c u = \mathcal{B}u$ *for all* $u \in D(A_c)$. *Then* $-A_c$ *is the generator of a* C_0*-semigroup in* $C(\overline{\Omega})$ *which is holomorphic with angle* $\frac{\pi}{2}$.

Using the arguments as in [War] we obtain a C_0-semigroup on $\mathbb{L}_1$ with optimal angle. For the convenience of the reader we give a direct proof.

Corollary 5.6. *Adopt the notation and assumptions as in Theorem 4.1. The semigroup generated by* $-\mathbb{A}$ *extends consistently to a* C_0*-semigroup on* $\mathbb{L}_1$ *which is holomorphic with angle* $\frac{\pi}{2}$.

Proof. Since the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ is of the same type as $\mathfrak{a}$ with α replaced by $\overline{\alpha}$, all the above is also valid for $\mathbb{A}^*$ instead of $\mathbb{A}$. Let $(\mathbb{A}^*)_c$ be the part of $\mathbb{A}^*$ in X_c.

Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Let $(u, \varphi) \in \mathbb{L}_2$ and $(v, \psi) \in X_c$. Then

$$\begin{aligned} |(e^{-z\mathbb{A}}(u,\varphi), (v,\psi))| &= |((u,\varphi), e^{-\overline{z}\mathbb{A}^*}(v,\psi))| \\ &= |((u,\varphi), e^{-\overline{z}(\mathbb{A}^*)_c}(v,\psi))| \le \|(u,\varphi)\|_{\mathbb{L}_1} \, \|e^{-\overline{z}(\mathbb{A}^*)_c}\|_{X_c \to X_c} \, \|(v,\psi)\|_{\mathbb{L}_\infty}. \end{aligned}$$

Hence $\|e^{-z\mathbb{A}}(u,\varphi)\|_{\mathbb{L}_1} \leq \|e^{-\overline{z}(\mathbb{A}^*)_c}\|_{X_c \to X_c} \|(u,\varphi)\|_{\mathbb{L}_1}$. By density the operator $e^{-z\mathbb{A}}$ extends to a continuous operator $S^{(1)}_z$ from $\mathbb{L}_1$ into $\mathbb{L}_1$ with norm $\|S^{(1)}_z\|_{\mathbb{L}_1\to\mathbb{L}_1} \leq \|e^{-\overline{z}(\mathbb{A}^*)_c}\|_{X_c\to X_c}$. It is easy to verify that $S^{(1)}_z \, S^{(1)}_w = S^{(1)}_{z+w}$ for all $z, w \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$. Also for all $\theta \in (0, \frac{\pi}{2})$ there are $M, \omega > 0$ such that $\|S^{(1)}_z\|_{\mathbb{L}_1\to\mathbb{L}_1} \leq M \, e^{\omega|z|}$ for all $z \in \Sigma_\theta$. Hence $z \mapsto \langle S^{(1)}_z(u,\varphi), (v,\psi)\rangle_{\mathbb{L}_1\times\mathbb{L}_\infty}$ is holomorphic on Σ°_θ for all $(v,\psi) \in \mathbb{L}_\infty$ and $(u,\varphi) \in \mathbb{L}_2$, but then also for all $(u,\varphi) \in \mathbb{L}_1$. Since the measure on $\Omega \oplus \Gamma$ is finite and $(e^{-z\mathbb{A}})_{z\in\Sigma^\circ_\theta}$ is continuous it follows that $(S^{(1)}_z)_{z\in\Sigma^\circ_\theta}$ is a weakly continuous semigroup on $\mathbb{L}_1$ for all $\theta \in (0, \frac{\pi}{2})$. Then the corollary follows using [Yos] Theorem IX.1. □

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